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CANARD HOMING ARTILLERY MODULAR PROJECTILE
(CHAMP)
PHASE II, EXPLORATORY DEVELOPMENT

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Final Technical Report Volume II

THEORY AND SIMULATION OF CONTROLLED PROJECTILES

February 1977

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Large Caliber Weapons Systems Laboratory
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details are provided for constructing a non-linear 7DOF simulation, employing the techniques of the Modular Software System, a Sanders developed simulation computer program that eliminates user involvement with software debugging.

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This volume supplements Vol. I by providing a theoretical basis for controlled spinning projectiles and their simulation. Included are equations of motion, coordinate transformations, aerodynamic forces and moments, and perturbation equations. A hierarchy of simulation models is developed starting with the full seven degrees of freedom (7DOF) model and including 4DOF and 5DOF models with time variant and time invariant parameters. The

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INTRODUCTION TO EQUATIONS OF MOTION FOR CONTROLLED SPINNING PROJECTILES

The notations and conventions which have been adopted are consistent with old NACA practice, with the report by Charters, reference [1] , and with the monograph by Jones, reference [2].

Newton's laws for describing the motion of rigid bodies state that the sum of all external forces acting on a body equals the time rate of change of momentum, and the sum of the moments of forces equals the time rate of change of moment of momentum.

$$\left.\begin{aligned} \Sigma\bar{F} &= \frac{d}{dt}(m\bar{U}) \\ \Sigma\bar{M} &= \frac{d}{dt}(\bar{H}) \end{aligned}\right\} \qquad (1\text{-}1)$$

All quantities in equations (1-1) are specified relative to axes fixed in space. Consequently, the six component equations of motion represented by the vector expressions of equations (1-1) are too unwieldy for practical use, since the moments of forces and the body inertia matrix would vary with time with respect to an inertial (fixed) axis system. The difficulty is overcome by referring all quantities to an Eulerian or moving axis system which is coincident with a set of axes fixed on the body. The formulation in terms of moving axes is made by recalling that:

$$\left.\frac{d}{dt}\right|_{\text{fixed axes}} = \left.\frac{d}{dt}\right|_{\text{moving axes}} + \bar{\Omega}\times$$

which permits equations (1-1) to be written as

$$\left.\begin{aligned} \Sigma\overline{F} &= \frac{d(m\overline{U})}{dt} + \overline{\Omega}x(m\overline{U}) \\ \Sigma\overline{M} &= \frac{d(\overline{H})}{dt} + \overline{\Omega}x(\overline{H}) \end{aligned}\right\} \qquad (1\text{-}2)$$

Equations (1-2) describe body motion in terms of linear momentum and moments of momentum referred to the moving coordinate, and Ω, the angular velocity vector of the moving frame relative to inertial (fixed) space.

The force components and the moment components are resolved onto a coordinate frame which is associated with the projectile.

There are a number of coordinate frames which are useful in the analysis of controlled spinning projectiles. These will be introduced at this time, even though most of them will not be used until much later. The justification for introducing them here is that most of this notational complexity is caused by the multiplicity of coordinate frames.

Each coordinate frame is a right-handed coordinate frame with origin O, and three mutually orthogonal axes which are designated the x-axis, y-axis, and z-axis, respectively. A right-handed coordinate frame is one in which the conventions for positive rotations have been coordinated with the positive direction of the axes so that a right-handed screw, pointed in the positive direction of an axis will advance in a positive direction when rotated in a clockwise sense, as seen looking in the positive direction. A right-handed frame is shown in Figure (1-1).

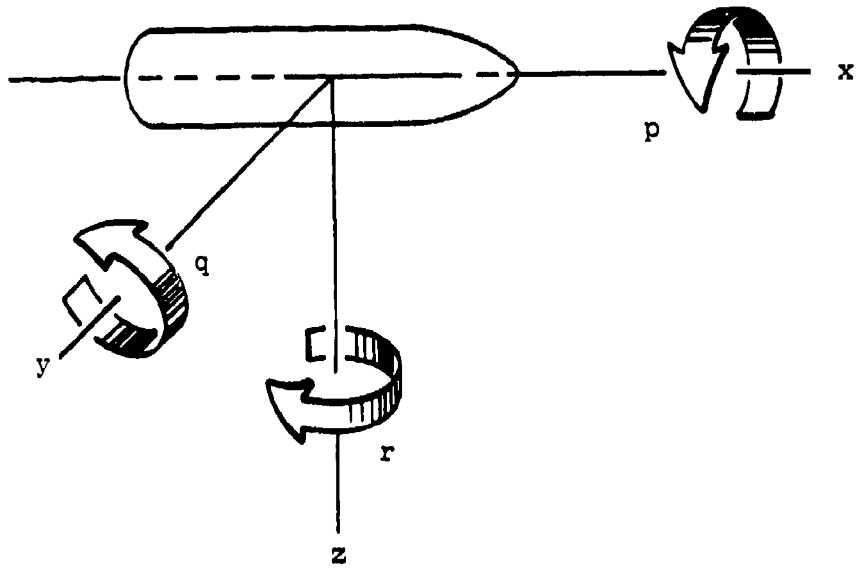


Figure (1-1). A Right-Handed Coordinate Frame

The following coordinate frames will be useful.

x^o y^o z^o	an inertial coordinate frame
x^F y^F z^F	a coordinate frame fixed to the cruciform canards
x' y' x'	a coordinate frame fixed in the body of the projectile
x'' y'' z''	a stabilized coordinate frame which moves with the projectile
x^T y^T z^T	a coordinate frame which is fixed in the target
x y z	an aeroballistic coordinate frame

Several vectors will be resolved onto the several coordinate frames as follows:

$$\left.\begin{aligned}
\bar{R} &= ix + jy + kz \; ; \quad \text{position} \\
\bar{U} &= iu + jv + kw \; ; \quad \text{velocity} \\
\bar{A} &= ia + jb + kc \; ; \quad \text{translational acceleration} \\
\bar{\Omega} &= ip + jq + kr \; ; \quad \text{angular velocity} \\
\dot{\bar{\Omega}} &= i\dot{p} + j\dot{q} + k\dot{r} \; ; \quad \text{angular acceleration} \\
\bar{F} &= iX + jY + kZ \; ; \quad \text{force} \\
\bar{M} &= iL + jM + kN \; ; \quad \text{moment}
\end{aligned}\right\} \qquad (1\text{-}3)$$

where i, j, and k are unit vectors in the x, y, z directions, respectively.

The equations of motion of a rigid body have been given by Jones. The equations are expressed with respect to a moving coordinate frame whose origin coincides with the center of gravity of the rigid body. The equations will be written initially with respect to an unprimed coordinate frame and the semantics of the equations will tighten up in the discussion which ensues. The equations are:

$$\left.\begin{aligned}
m[\dot{u} - vr + wq] &= X + X_G \\
m[\dot{v} - wp + ur] &= Y + Y_G \\
m[\dot{w} - uq + vp] &= Z + Z_G
\end{aligned}\right\} \quad \text{Force Equations} \qquad (1\text{-}4)$$

$$\left.\begin{aligned}
\dot{h}_1 - h_2 r + h_3 q &= L \\
\dot{h}_2 - h_3 p + h_1 r &= M \\
\dot{h}_3 - h_1 q + h_2 p &= N
\end{aligned}\right\} \quad \text{Moment Equations} \qquad (1\text{-}5)$$

$$\left.\begin{aligned} h_1 &= Ap - Fq - Er \\ h_2 &= Bq - Dr - Fp \\ h_3 &= Cr - Ep - Dq \end{aligned}\right\} \quad \text{Angular Momentum Equations} \qquad (1\text{-}6)$$

where

X_G, Y_G, Z_G	are components of a force vector due to gravity
h_1, h_2, h_3	are components of an angular momentum vector
A,B,C,D,E,F	are components of an inertial tensor, and
A, B, C	are moments of inertia
D, E, F	are products of inertia

The application of equations (1-4), (1-5), and (1-6) imposes certain practical constraints. First, it is desirable to choose the coordinate frame so that the components of the inertial tensor (A, B, C, D, E, F) are constant. This can be accomplished for a projectile of constant mass by causing the coordinate frame to be fixed in the body of the projectile. This is the primed coordinate frame x', y', x'). Writing equation (1-4) relative to the primed coordinate frame, gives

$$\left.\begin{aligned} m[\dot{u}' - v'r' + w'q'] &= X' + X_G' \\ m[\dot{v}' - w'p' + u'r'] &= Y' + Y_G' \\ m[\dot{w}' - u'q' + v'p'] &= Z' + Z_G' \end{aligned}\right\} \quad \begin{array}{l}\text{Force Equations} \\ \text{Relative to} \\ \text{Body Frame}\end{array} \qquad (1\text{-}7)$$

The angular momentum equations, (1-6), become

$$\begin{bmatrix} h_1' \\ h_2' \\ h_3' \end{bmatrix} = \begin{bmatrix} A' & -F' & -E' \\ -F' & B' & -D' \\ -E' & -D' & C' \end{bmatrix} \begin{bmatrix} p' \\ q' \\ r' \end{bmatrix} \tag{1-8}$$

where the coefficients A'. B', C', D', E', F' are constants. Differentiating (1-8) gives

$$\begin{bmatrix} \dot{h}_1' \\ \dot{h}_2' \\ \dot{h}_3' \end{bmatrix} = \begin{bmatrix} A' & -F' & -E' \\ -F' & B' & -D' \\ -E' & -D' & C' \end{bmatrix} \begin{bmatrix} \dot{p}' \\ \dot{q}' \\ \dot{r}' \end{bmatrix} \tag{1-9}$$

Let the matrix be inverted so that

$$E = \begin{bmatrix} A' & -F' & -E' \\ -F' & B' & -D' \\ -E' & -D' & C' \end{bmatrix} \; ; \quad E^{-1} = \begin{bmatrix} A' & -F' & -E' \\ -F' & B' & -D' \\ -E' & -D' & C' \end{bmatrix}^{-1} \tag{1-10}$$

Then,

$$\dot{\bar{\Omega}}' = \begin{bmatrix} \dot{p}' \\ \dot{q}' \\ \dot{r}' \end{bmatrix} = E^{-1} \begin{bmatrix} \dot{h}_1' \\ \dot{h}_2' \\ \dot{h}_3' \end{bmatrix} \tag{1-11}$$

Combining (1-11) and (1-5)

$$\begin{bmatrix} \dot{p}' \\ \dot{q}' \\ \dot{r}' \end{bmatrix} = E^{-1} \begin{bmatrix} L' + h_2'r' - h_3'q' \\ M' + h_3'p' - h_1'r' \\ N' + h_1'q' - h_2'p' \end{bmatrix} \tag{1-12}$$

Introducing the notation

$$\left.\begin{aligned} a' &= X'/m \\ b' &= Y'/m \\ c' &= Z'/m \end{aligned}\right\} \tag{1-13}$$

The equations are:

$$\begin{aligned} \dot{u}' &= v'r' - w'q' + a' + (X_G'/m) \\ \dot{v}' &= w'p' - u'r' + b' + (Y_G'/m) \\ \dot{w}' &= u'q' - v'p' + c' + (Z_G'/m) \end{aligned} \tag{1-14}$$

$$\begin{bmatrix} \dot{p}' \\ \dot{q}' \\ \dot{r}' \end{bmatrix} = E^{-1} \begin{bmatrix} L' + h_2'r' - h_3'q' \\ M' + h_3'p' - h_1'r' \\ N' + h_1'q' - h_2'p' \end{bmatrix} \tag{1-15}$$

Equations (1-13), (1-14), and (1-15) are the most general nonlinear equations of motion which will be considered.

Second, this most general case can be constrained by imposing a rotational symmetry about the x' axis. The full significance of this symmetry can best be implied in terms of the inertial coefficients. The inertial coefficients are defined as follows (see reference [3], p. 139).

$$\left.\begin{aligned}
A' &= \int_\tau (y'^2 + z'^2)\rho d\tau \\
B' &= \int_\tau (z'^2 + x'^2)\rho d\tau \\
C' &= \int_\tau (x'^2 + y'^2)\rho d\tau \\
D' &= \int_\tau y'z' \ \rho d\tau \\
E' &= \int_\tau z'x' \ \rho d\tau \\
F' &= \int_\tau x'y' \ \rho d\tau
\end{aligned}\right\} \qquad (1\text{-}16)$$

where

ρ = density

$d\tau$ = volume increment

Let the x',y',z' axes (the primed coordinate frame) be aligned initially to coincide with the x,y,z axes. Then, let the x' axis and the x axis remain coincident while the primed coordinate frame is spun about the coincident axes. The inertial components about the unprimed axes will generally not be constant. However, they will be constant provided the coincident spin axes; i.e.,x' axis and x axis, are axes of summetry. Under these conditions

$$\left.\begin{aligned}
&A \text{ is constant} \\
&B = C \\
&D = E = F = 0
\end{aligned}\right\} \qquad (1\text{-}17)$$

When these axial symmetry conditions are satisfied, the equations of motion can be written in the following form:

$$\left.\begin{aligned}
\dot{u} &= vr - wq + a + (X_G/m) \\
\dot{v} &= wp - ur + b + (Y_G/m) \\
\dot{w} &= uq - vp + c + (Z_G/m)
\end{aligned}\right\} \qquad (1\text{-}18)$$

$$\left.\begin{aligned} h_1 &= Ap' \\ h_2 &= Bq \\ h_3 &= Br \end{aligned}\right\} \qquad (1\text{-}19)$$

$$\left.\begin{aligned} \dot{p}' &= (L/A) \\ \dot{q} &= (M + h_3 p - h_1 r)/B \\ \dot{r} &= (N + h_1 q - h_2 p)/B \end{aligned}\right\} \qquad (1\text{-}20)$$

Note that the angular momentum component (h_1) depends on the spin rate of the body (p') and is independent of the spin rate of the unprimed coordinate frame (p).

The usefulness of equations (1-18) through (1-20) depends on whether the force components (X, Y, Z) and the moment components (L, M, N) can be expressed conveniently in the unprimed coordinate frame. They can be and the manner of doing so has been described by Charters [1], pp. 27-33. According to Charters:

> "... It may be concluded, therefore, that the aerodynamic coefficients are invariant with respect to rotation of coordinate axes about the axis of symmetry. Consequently, the y and z axes may be oriented at will around the x axis, without regard to the orientation of the missile about x, since the aerodynamic coefficients are solely functions of the missile's external shape (and such non-dimensional parameters as Reynolds and Mach numbers) and it has been shown that the aerodynamic coefficients do not change with orientation of the missile in roll. In fact, the missile may be allowed to spin about the x axis with respect to the y,z axes and the aerodynamic coefficients will be unaffected."

Charters defines three sets of coordinate axes. One set, which he calls "earth axes" correspond to the inertial coordinate frame, x^o, y^o, z^o. A set, which he calls body axes, corresponds to our coordinate frame fixed in the body, x', y', z'. A set, which he calls "pseudo-stability axes" corresponds thus far to the as yet unnamed unprimed coordinate frame, x, y, z. Charters points out that the pseudo-stability axes may be oriented at will about the axis of rotational symmetry. The pseudo-stability axes are commonly referred to as aeroballistic axes.

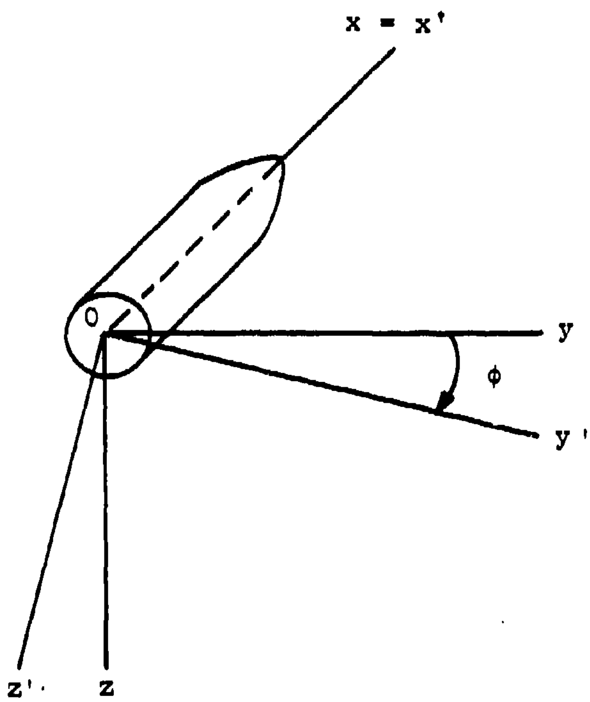
It should be pointed out that Charters pseudo-stability axes are not associated with a physical body. Thus, the magnitude of the angular velocity about the x axis can be specified arbitrarily and does not have to satisfy an equation of dynamic equilibrium.

A coordinate frame will be associated with the canard assembly, or fin-frame, of a controlled spinning projectile. This will require an equation of dynamic equilibrium which is introduced in the following paragraphs.

The geometrical relationship between the primed and unprimed coordinate frames is the same whether or not the unprimed frame is physical. This relationship is shown in Figure (1-2).

One useful treatment of the unprimed coordinate frame is to set its spin rate (p) equal to zero. The motion of the projectile is then described relative to a non-spinning coordinate frame and the equations are

$$\left.\begin{aligned} \dot{u} &= vr - uq + a + (X_G/m) \\ \dot{v} &= -ur + b + (Y_G/m) \\ \dot{w} &= uq + c + (Z_G/m) \end{aligned}\right\} \qquad (1\text{-}21)$$



(1) The primed coordinate frame is fixed in the body.

(2) The origins are coincident.

(3) The Ox and Ox' axes are coincident.

$$\dot{\phi} = p' - p$$

$$\phi = \phi(0) + \int_0^t (p'-p)dt$$

$$a = a_0'$$

$$b = b_0' \cos\phi - c_0' \sin\phi$$

$$c = b_0' \sin\phi + c_0' \cos\phi$$

$$q = q'\cos\phi - r'\sin\phi$$

$$r = q'\sin\phi + r'\cos\phi$$

Figure 1-2. The Relationship between the Primed and Unprimed Coordinate Frames

$$\left.\begin{aligned} p &= 0 \\ \dot{p}_B' &= L/A \\ \dot{q} &= (M - Ap'r)/B \\ \dot{r} &= (N + Ap'q)/B \end{aligned}\right\} \qquad (1\text{-}22)$$

In order to associate the unprimed coordinate frame with the aeroballistic frame, first the spin angular momentum is defined as

$$h_1 = A_B p_B' + A_F p_F \qquad (1\text{-}23)$$

It is assumed that the fin-frame will have axial symmetry, just as the body does, and that its products of inertia are zero. Equating the rate-of-change of the spin momentum to a total rolling moment,

$$\dot{h}_1 = A_B \dot{p}_B' + A_F \dot{p}_F = L \qquad (1\text{-}24)$$

However, another physical variable is required and this is the torque, T, between the body frame and the fin-frame. Let

$$\left.\begin{aligned} A_F \dot{p}_F &= L_F + T \\ A_B \dot{p}_B' &= L_B - T \end{aligned}\right\} \qquad (1\text{-}25)$$

where

L_F = aerodynamic moment applied to the fin-frame

L_B = aerodynamic moment applied to the body-frame.

It can be seen that equations (1-25) satisfy equation (1-24), provided that

$$L = L_F + L_B \qquad (1\text{-}26)$$

The equations which have been developed here are summarized in Table 1. Three sets of equations have been identified:

(1) Six degrees-of-freedom expressed in body axes

(2) Six degrees-of-freedom expressed in aeroballistic (pseudo-stability) axes

(3) Seven degrees-of-freedom expressed in aeroballistic axes

The first set is most general but it presents a number of algebraic and computational difficulties. The second set introduces the constraint of axial symmetry and a number of approximations which are convention for aeroballistics. However, it does not include an explicit description of the stabilized and controllable fin-frame. This second set has been included here in order to show a relationship to conventional aeroballistic theory. The third set of equations introduces a seventh degree-of-freedom: the spin of the fin-frame independent of the body. This seven degree-of-freedom model will be the basis for most of the theoretical design considerations for controlled spinning projectiles.

TABLE 1-1. ALTERNATIVE FORMULATIONS OF EQUATIONS OF MOTION

6 DOF Equations of Motion Expressed in Body Axes	6 DOF Equations of Motion Expressed in Pseudo-Stability Axes	7 DOF Equations of Motion Expressed in Pseudo-Stability Axes
$a' = X'/m$	$a = X/m$	$a = X/m$
$b' = Y'/m$	$b = Y/m$	$a = Y/m$
$c' = Z'/m$	$c = Z/m$	$c = Z/m$
$\dot{u}' = v'r' - w'q' + a' + (X'/m)$ $\dot{v}' = w'p' - u'r' + b' + (Y'/m)$ $\dot{w}' = u'q' - v'p' + c' + (Z'/m)$	$\left.\begin{aligned}\dot{u} &= vr - wq + a + (X_G/m)\\ \dot{v} &= -ur + b + (Y_G/m)\\ \dot{w} &= uq + c + (Z_G/m)\end{aligned}\right\}$	$\left.\begin{aligned}\dot{u} &= vr - wq + a + (X_G/m)\\ \dot{v} &= -ur + b + (Y_G/m)\\ \dot{w} &= uq + c + (Z_G/m)\end{aligned}\right\}$
$\mathbf{E} = \begin{bmatrix} A' & -F' & -E' \\ -F' & B' & -D' \\ -E' & -D' & C' \end{bmatrix}; \begin{bmatrix} h_1' \\ h_2' \\ h_3' \end{bmatrix} + \mathbf{E}\begin{bmatrix} p' \\ q' \\ r' \end{bmatrix}$	$\left.\begin{aligned}h_1 &= Ap'\\ h_2 &= Bq\\ h_3 &= Br\end{aligned}\right\}$ NOTE: Axial symmetry is required	$\left.\begin{aligned}h_1 &= A_B p_B' + A_F p_F\\ h_2 &= Bq\\ h_3 &= Br\end{aligned}\right\}$ NOTE: Axial symmetry is required
$\begin{bmatrix} \dot{p}' \\ \dot{q}' \\ \dot{r}' \end{bmatrix} = \mathbf{E}^{-1}\begin{bmatrix} L' + h_2'r' - h_3'q' \\ M' + h_2'p' - h_1'r' \\ N' + h_1'q - h_2'p' \end{bmatrix}$	$\left.\begin{aligned}\dot{p}' &= L/A\\ \dot{q} &= (M - Ap'r)/B\\ \dot{r} &= (N + Ap'q)/B\end{aligned}\right\}$	$\left.\begin{aligned}\dot{p}_B' &= (L_B - T)/A_B\\ \dot{p}_F &= (L_F + T)/A_F\end{aligned}\right\}$ $\left.\begin{aligned}\dot{q} &= (M - h_1 r)/B\\ \dot{r} &= (N + h_1 q)/B\end{aligned}\right\}$

SECTION 2

COORDINATE TRANSFORMATIONS

Three dimensional Cartiesian vectors transform according to the scheme:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} u^o \\ v^o \\ w^o \end{bmatrix} ; \qquad (2\text{-}1)$$

The inverse transformation is:

$$\begin{bmatrix} u^o \\ v^o \\ w^o \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} ; \qquad (2\text{-}2)$$

and the transpose of the matrix is also its inverse. The matrix elements, c_{ij}, are called "direction cosines" and are capable of physical interpretation.

The properties of these coordinate transformations are most familiar to engineers in terms of Euler angles (gimbal angles). Such a presentation is given in Charters [1], and none will be repeated here. A less familiar treatment of the coordinate transformations in terms of the components of a rotation vector is given in Webster [4] and will be discussed here.

The rates-of-change of the direction cosines depend on the present values of the direction cosines and components of angular velocity. The complete relationships are:

$$\left.\begin{aligned}
\dot{c}_{11} &= c_{21}r - c_{31}q \\
\dot{c}_{12} &= c_{22}r - c_{32}q \\
\dot{c}_{13} &= c_{23}r - c_{33}q \\
\\
\dot{c}_{21} &= c_{31}p - c_{11}r \\
\dot{c}_{22} &= c_{32}p - c_{12}r \\
\dot{c}_{23} &= c_{33}p - c_{13}r \\
\\
\dot{c}_{31} &= c_{11}q - c_{21}p \\
\dot{c}_{32} &= c_{12}q - c_{22}p \\
\dot{c}_{33} &= c_{13}q - c_{23}p
\end{aligned}\right\} \qquad (2\text{-}3)$$

Each rate-of-change must be integrated:

$$c_{ij} = c_{ij}(0) + \int_0^t \dot{c}_{ij}\, dt \qquad (2\text{-}4)$$

The physical significance of the direction cosines can be demonstrated as follows. Consider a unit vector in the direction of the x^o-axis. It can be expressed as a column vector:

$$\begin{bmatrix} x^o \\ y^o \\ z^o \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (2\text{-}5)$$

This vector will transform onto the unprimed coordinate frame in accordance with equation (2-1).

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} \qquad (2\text{-}6)$$

Thus, c_{11} is the cosine of the angle between the x^o-axis and the x-axis; c_{21} the cosine of the angle between the x^o-axis and the y-axis; and c_{31} is the cosine of the angle between the x^o-axis and z-axis.

The argument can be repeated using vectors

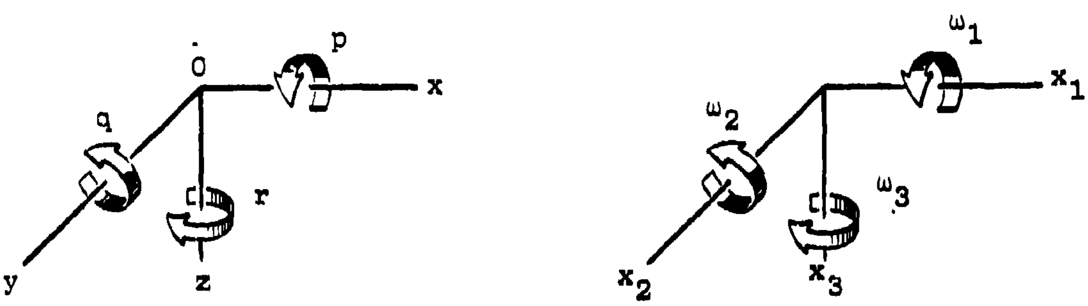
$$\begin{bmatrix} x^o \\ y^o \\ z^o \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \text{a unit vector along the } y^o\text{-axis}$$

$$\begin{bmatrix} x^o \\ y^o \\ z^o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \text{a unit vector along the } z^o\text{-axis}$$

in order to interpret the significance of the remaining c_{ij}.

We have chosen a mnemonic notation for the vector components because we have found that it facilitates learning and remembering the notation. However, there is some merit in a pure index notation and such a notation will facilitate the present discussion.

The equivalence of a mnemonic notation and an index notation for a right-handed coordinate frame is shown in Figure (2-1). The axis are assocated with the indices 1, 2, and 3 and the same symbol is employed for each axis or for each vector component. The same symbols are employed for each unit vector and the unit vectors are distinguished from one another by the numerical indices.



$\overline{R} = ix + jy + kz$

$\overline{\Omega} = ip + jq + kr$

(ijk are unit vectors)

$\overline{X} = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$

$\overline{\Omega} = \omega_1\varepsilon_1 + \omega_2\varepsilon_2 + \omega_3\varepsilon_3$

($\varepsilon_1\varepsilon_2\varepsilon_3$ are unit vectors)

MNEMONIC NOTATION

INDEX NOTATION

Figure (2-1). Equivalence of Mnemonic Notation and Index Notation for a Right-Handed Coordinate Frame

The rates-of-change of the direction cosines can be expressed in the index notation:

$$\left.\begin{aligned}
\dot{c}_{11} &= c_{21}\omega_3 - c_{31}\omega_2 \\
\dot{c}_{12} &= c_{22}\omega_3 - c_{32}\omega_2 \\
\dot{c}_{13} &= c_{33}\omega_3 - c_{33}\omega_2 \\
\\
\dot{c}_{21} &= c_{31}\omega_2 - c_{11}\omega_1 \\
\dot{c}_{22} &= c_{32}\omega_2 - c_{12}\omega_1 \\
\dot{c}_{23} &= c_{33}\omega_2 - c_{13}\omega_1 \\
\\
\dot{c}_{31} &= c_{11}\omega_2 - c_{21}\omega_1 \\
\dot{c}_{32} &= c_{12}\omega_2 - c_{22}\omega_1 \\
\dot{c}_{33} &= c_{13}\omega_2 - c_{23}\omega_1
\end{aligned}\right\} \qquad (2\text{-}7)$$

Thus, the general expression is:

$$\dot{c}_{ki} = c_{k+1,i}\omega_{k-1} - c_{k-1,i}\omega_{k+1} \qquad (2\text{-}8)$$

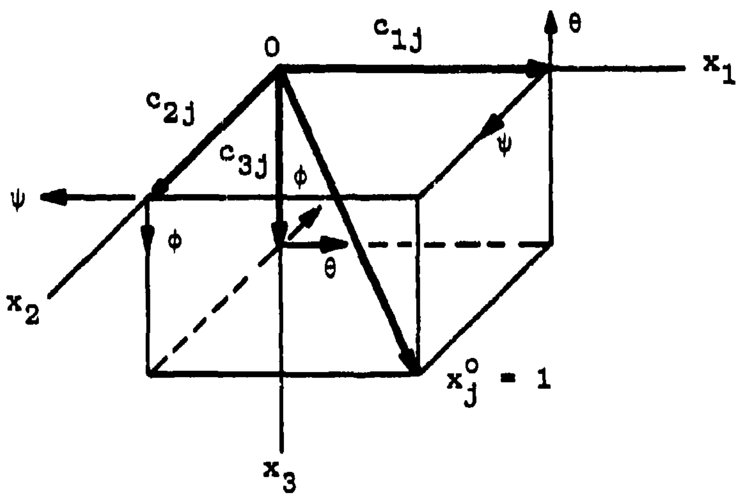
where k+1 and k-1 are to be evaluated with respect to the following truth table

k	k+1	k-1
1	2	3
2	3	1
3	1	2

The physical basis for equations (2-3) or (2-7) can be demonstrated by considering small rotations instead of rates-of-change. That is

$$
\begin{aligned}
\phi &= \int p\,dt = \int \omega_1 dt \\
\theta &= \int q\,dt = \int \omega_2 dt \qquad (2\text{-}9) \\
\psi &= \int r\,dt = \int \omega_3 dt
\end{aligned}
$$

The effect of these small rotations is to change the projections of a unit vector in the inertial frame, $x_j^o = 1$, on the axes of the unprimed frame. The effect of these changes is depicted in Figure 2-2.



The Unprimed Frame Rotates Through Small Angles ϕ, θ, ψ.

Figure 2-2. Physical Basis for the Change in Direction Cosines Caused by Small Rotations.

and their magnitude can be written by inspection. Thus

$$\left.\begin{aligned} \Delta C_{1j} &= C_{2j}\psi - C_{3j}\theta \\ \Delta C_{2j} &= C_{3j}\phi - C_{1j}\psi \\ \Delta C_{3j} &= C_{1j}\theta - C_{2j}\phi \end{aligned}\right\} \tag{2-10}$$

The several vectors described in Equations 1-3 can be transformed from one coordinate frame to another according to the following schemes:

$$\begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 2_{33} \end{bmatrix} \begin{bmatrix} u^o \\ v^o \\ w^o \end{bmatrix}; \quad \bar{U}^T = \mathbf{A}\,\bar{U}^o \tag{2-11}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} u^o \\ v^o \\ w^o \end{bmatrix}; \quad \bar{U}^1 = \mathbf{B}\,\bar{U}^o \tag{2-12}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} u^o \\ v^o \\ w^o \end{bmatrix}; \quad \bar{U} = \mathbf{C}\,\bar{U}^o \tag{2-13}$$

$$\begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix} \quad \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}; \quad \bar{U}'' = \mathbf{D}\,\bar{U} \tag{2-14}$$

SECTION 3

THE SEVEN DEGREE-OF-FREEDOM MODEL

The seven degree-of-freedom model for a controlled spinning projectile has been derived in Section 1. However, it will be helpful to restate the equations here:

$$\left.\begin{aligned} a &= X/m \\ b &= Y/m \\ c &= Z/m \end{aligned}\right\} \text{Accelerometer Equations} \qquad (3\text{-}1)$$

$$\left.\begin{aligned} \dot{u} &= vr - wq + a + (X_G/m) \\ \dot{v} &= -ur + b + (Y_G/m) \\ \dot{w} &= uq + c + (Z_G/m) \end{aligned}\right\} \text{Force Equalibruim Equations} \qquad (3\text{-}2)$$

$$\left.\begin{aligned} h_1 &= A_B p_B' + A_F p_F \\ h_2 &= Bq \\ h_3 &= Br \end{aligned}\right\} \text{Angular Momentum Equations} \qquad (3\text{-}3)$$

$$\left.\begin{aligned} \dot{p}_B' &= (L_B - T)/A_B \\ \dot{p}_F &= (L_F + T)/A_F \\ \dot{q} &= (M - h_1 r)/B \\ \dot{r} &= (N + h_1 q)/B \end{aligned}\right\} \text{Moment Equilibrium Equations} \qquad (3\text{-}4)$$

These equations are expressed relative to the aeroballistic frame. The seven degrees of freedom are u, v, w, p_F, p_B', q and r.

A number of details about the seven degree-of-freedom model must be clarified. These include:

(1) Transformation of the velocity components to the inertial frame from the aeroballistic frame.

(2) Transformation of the gravity components to the aeroballistic frame.

(3) Evaluation of the aerodynamic forces and moments (X, Y, Z, L_F, L_B, M, N).

Equations (3-1) through (3-4) describe the motion of a controlled spinning projectile with respect to the aeroballistic frame. The orientation of the aeroballistic frame in inertial space is determined by the angular velocity vector ($\bar{\Omega}$) where

$$\bar{\Omega} = \begin{bmatrix} 0 \\ q \\ r \end{bmatrix} \tag{3-5}$$

These vector components are used to generate the transformation matrix as follows:

$$\left.\begin{aligned}
\dot{c}_{11} &= c_{21}r - c_{31}q \\
\dot{c}_{12} &= c_{22}r - c_{32}q \\
\dot{c}_{13} &= c_{23}r - c_{33}q \\[1ex]
\dot{c}_{21} &= -c_{11}r \\
\dot{c}_{22} &= -c_{12}r \\
\dot{c}_{23} &= -c_{13}r \\[1ex]
\dot{c}_{31} &= c_{11}q \\
\dot{c}_{32} &= c_{12}q \\
\dot{c}_{33} &= c_{13}q
\end{aligned}\right\} \tag{3-6}$$

and these derivatives are integrated as follows

$$c_{ij} = c_{ij}(0) + \int_0^t \dot{c}_{ij}\, dt \qquad (3\text{-}7)$$

The transformation matrix is used to transform the u, v, w components onto the inertial frame:

$$\bar{U}^{\circ} = \mathbf{C}^{-1}\,\bar{U}$$

$$\begin{bmatrix} u^{\circ} \\ v^{\circ} \\ w^{\circ} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad (3\text{-}8)$$

Components of wind aloft may be transformed onto the aeroballistic frame as follows:

$$\bar{U}_W = \mathbf{C}\,\bar{U}_W^{\circ}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} u_W^{\circ} \\ v_W^{\circ} \\ w_W^{\circ} \end{bmatrix} \qquad (3\text{-}9)$$

Also, the weight of the projectile can be transformed from the inertial frame to the aeroballistic frame

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} \quad \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} \qquad (3\text{-}10)$$

It will be convenient to divide equation (3-10) by m and to let

$$\left.\begin{aligned} g &= W/m \\ g_x &= X_G/m \\ g_y &= Y_G/m \\ g_z &= Z_G/m \end{aligned}\right\} \qquad (3\text{-}11)$$

then

$$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \qquad (3\text{-}12)$$

Consistent with equation (3-12) it will be convenient to rewrite equation (3-2) as:

$$\left.\begin{aligned} \dot{u} &= vr - wq + a + g_x \\ \dot{v} &= -ur + b + g_y \\ \dot{w} &= uq + c + g_z \end{aligned}\right\} \qquad (3\text{-}13)$$

The velocity components, u^o, v^o, w^o, are integrated to obtain position coordinates as follows:

$$\left.\begin{aligned} x^o &= x^o(0) + \int_0^t u^o dt \\ y^o &= y^o(0) + \int_0^t v^o dt \\ z^o &= z^o(0) + \int_0^t w^o dt \end{aligned}\right\} \qquad (3\text{-}14)$$

The aerodynamic forces and moments are expressed in the following forms

$$F = \frac{\rho V^2 S}{2} C$$

$$M = \frac{\rho V^2 S d}{2} C$$

where

F = force component

M = moment component

C = dimensionless coefficient

$\frac{1}{2}\rho V^2$ = dynamic pressure

S = reference area

d = reference length

The complete sets of forces and moments which are required are

$$\left.\begin{aligned} X &= (\rho V^2 S/2)C_X \\ Y &= (\rho V^2 S/2)C_Y \\ Z &= (\rho V^2 S/2)C_Z \end{aligned}\right\} \quad (3\text{-}15)$$

$$\left.\begin{aligned} L_F &= (\rho V^2 Sd/2)C_{LF} \\ L_B &= (\rho V^2 Sd/2)C_{LB} \\ M &= (\rho V^2 Sd/2)C_M \\ N &= (\rho V^2 Sd/2)C_N \end{aligned}\right\} \quad (3\text{-}16)$$

where X, Y, Z are forces in the x, y, z directions, respectively, and L, M, N are moments about the x, y, z directions, respectively.

The moments are described by the adjectives rolling, pitching, and yawing, respectively, but no similar adjectives have been developed for the forces.

The subscripts, $(\ )_B$ and $(\ )_F$, have been introduced to distinguish between the rolling moments which acts on the body (B) and the rolling moments which acts on the fin frame (F). The corresponding moment coefficients are C_{LB} and C_{LF}, respectively.

It became common practice to expand the dimensionless aerodynamic coefficients as a series expressed in terms of a number of dimensionless variables. The more important dimensionless variables are:

α	angle-of-attack (rad)
β	side-slip angle (rad)
$(p_B d/2V)$	dimensionless roll-rate of the body (rad/sec)
$(p_F d/2V)$	dimensionless roll-rate of the fin frame, (rad/sec)
$(qd/2V)$	dimensionless pitch-rate
$(rd/2V)$	dimensionless yaw-rate
δ	control deflection (rad)
Δ	cant angle (rad)

The coefficients of each series are partial derivatives of the dimensionless coefficients with respect to the dimensionless variables.

However, it has not become common practice to employ a partial derivative notation. Instead, a notation scheme with subscripts is employed. The following conventions will be adopted for specifying the non-dimensional aerodynamic coefficients ([1], p.23).

$$
\left.\begin{aligned}
C_X &= -C_A \\
C_Y &= C_{Y\beta}\ \beta + C_{Y\delta}\ \delta_Z \\
C_Z &= -C_{Z\alpha}\ \alpha - C_{Z\delta}\ \delta_Y \\
C_{LF} &= C^F_{\ell p}\left(\frac{p_F d}{2V}\right) + C^F_{\ell\Delta}\ \Delta_F + C^F_{\ell\beta\delta}\ \beta\ \delta_Y + C^F_{\ell\alpha\delta}\ \alpha\ \delta_Z \\
C_{LB} &= C^B_{\ell p}\left(\frac{p_B' d}{2V}\right) + C^B_{\ell\Delta}\ \Delta_B \\
C_M &= C_{m\alpha}\ \alpha + C_{mq}\left(\frac{qd}{2V}\right) + C_{m\beta p}\ \beta\left(\frac{p_B' d}{2V}\right) + C_{m\delta}\ \delta_Y \\
C_N &= C_{n\beta}\ \beta + C_{mr}\left(\frac{rd}{2V}\right) + C_{n\alpha p}\ \alpha\left(\frac{p_B' d}{2V}\right) + C_{n\delta}\ \delta_Z
\end{aligned}\right\} \quad (3\text{-}17)
$$

The coefficients of the dimensionless terms on the right hand sides of the equations are called aerodynamic derivatives and are dimensionless. The following relationships are valid because of rotational symmetry.

$$
\left.\begin{aligned}
C_{Y\beta} &= -C_{Z\alpha} \\
C_{Y\delta} &= C_{Z\delta} \\
C_{n\beta} &= -C_{m\alpha} \\
C_{nr} &= C_{mq} \\
C_{n\alpha p} &= C_{m\beta p} \\
C_{n\delta} &= C_{m\delta} \\
C^F_{\ell\beta\delta} &= C^F_{\ell\alpha\delta}
\end{aligned}\right\} \quad (3\text{-}18)
$$

Thus, a total of 19 aerodynamic derivatives of which 12 are independent will be used. The independent derivatives are listed below, along with an alternative notation.

C_x	CX	axial force
$C_{z\alpha}$	CZA	normal force due to angle-of-attack
$C_{z\delta}$	CZΔ	normal force due to control deflection
$C_{m\alpha}$	CMA	moment due to angle-of-attack
C_{mq}	CMQ	damping moment
$C_{n\alpha p}$	CNAP	magnus pitching moment
$C_{m\delta}$	CMΔ	moment due to control deflection
$C^F_{\ell p}$	CLPF	fin rolling moment due to rate-of-roll
$C^F_{\ell\Delta}$	CLΔF	fin rolling moment due to cant angle
$C^B_{\ell p}$	CLPB	body rolling moment due to rate-of-roll
$C^F_{\ell\beta\delta}$	CLBF	dihedral effect
$C^B_{\ell\Delta}\,\Delta_B$	CLΔB	body rolling moment due to cant angle

The symbols in the second column are mnemonics which may be useful in labelling computer printouts. (The derivatives have been listed here in the order in which they are introduced in simulation modules.) The expressions for the aerodynamic derivatives become:

$$\left.\begin{aligned}
C_x &= -C_A \\
C_y &= -C_{z\alpha}\,\beta + C_{z\delta}\,\delta_z \\
C_z &= -C_{z\alpha}\,\alpha - C_{z\delta}\,\delta_y \\
C_{LF} &= C^F_{\ell p}\left(\frac{p_F d}{2V}\right) + C^F_{\ell\Delta}\,\Delta_F + C^F_{\ell\beta\delta}\,\beta\,\delta_y + C^F_{\ell\beta\delta}\,\alpha\delta_z \\
C_{LB} &= C^B_{\ell p}\left(\frac{p'_B d}{2V}\right) + C^B_{\ell\Delta}\,\Delta_B \\
C_M &= C_{m\alpha}\,\alpha + C_{mq}\left(\frac{qd}{2V}\right) + C_{n\alpha p}\,\beta\left(\frac{p'_B d}{2V}\right) + C_{m\delta}\,\delta_y \\
C_N &= -C_{m\alpha}\,\beta + C_{mq}\left(\frac{rd}{2V}\right) + C_{n\alpha p}\,\alpha\left(\frac{p'_B d}{2V}\right) + C_{m\delta}\,\delta_z
\end{aligned}\right\} \tag{3-19}$$

The dimensionless aerodynamic derivatives are functions of Mach Number, M. For the present these functions will be treated as second degree polynomials, e.g.,

$$C_x = a_0 + a_1 M + a_2 M^2 \tag{3-20}$$

The following equations, express V, α and β in terms of the state variables u, v, and w, and the wind's components u_w, v_w and w_w.

$$\left.\begin{aligned} V^2 &= (u + u_w)^2 + (v + v_w)^2 + (w + w_w)^2 \\ \beta &= \tan^{-1} \frac{v + v_w}{u + u_w} \\ \alpha &= \tan^{-1} \frac{w + w_w}{\sqrt{(u + u_w)^2 + (v + v_w)^2}} \end{aligned}\right\} \tag{3-21}$$

The most important equations for the seven degree-of-freedom model are summarized in Table 3-1.

Fin deflection signs refer to those panels whose rotation axis when aligned with the positive y or z axis and are consistent with positive rotations about these axes. This applies to either panels which deflect together (δ) or panels which deflect in opposition (Δ).

TABLE 3-1

SUMMARY OF EQUATIONS FOR THE SEVEN DEGREE-OF-FREEDOM MODEL

Equations	
$a = X/m$ $b = Y/m$ $c = Z/m$	Accelerometer Equations
$\dot{u} = vr - wq + a + g_x$ $\dot{v} = -ur + b + g_y$ $\dot{w} = uq + c + g_z$	Force Equilibrium Equations
$h_1 = A_B p'_B + A_F p_F$ $h_2 = Bq$ $h_3 = Br$	Angular Momentum Equations
$\dot{p}'_B = (L_B - T)/A_B$ $\dot{p}_F = (L+T)/A_F$ $\dot{q} = (M - h_1 r)/B$ $\dot{r} = (N + h_1 q)/B$	Moment Equilibrium Equations
$X = (\rho V^2 S/2)C_x$ $Y = (\rho V^2 S/2)C_y$ $Z = (\rho V^2 S/2)C_z$ $L_F = (\rho V^2 Sd/2)C_{LF}$ $L_B = (\rho V^2 Sd/2)C_{LB}$ $M = (\rho V^2 Sd/2)C_M$ $N = (\rho V^2 Sd/2)C_N$	Aerodynamic Force & Moment Equations

TABLE 3-1

SUMMARY OF EQUATIONS FOR THE SEVEN DEGREE-OF-FREEDOM MODEL (Continued)

$$\left.\begin{aligned}
C_x &= -C_A \\
C_y &= -C_{z\alpha}\,\beta + C_{z\delta}\,\delta_z \\
C_z &= -C_{z\alpha}\,\alpha - C_{z\delta}\,\delta_y \\
C_{LF} &= C^F_{\ell p}\left(\frac{p_F d}{2V}\right) + C^F_{\ell\Delta}\,\Delta_F + C^F_{\ell\beta\delta}\,\beta\,\delta_y + C_{\ell\beta\delta}\,\alpha\delta_z \\
C_{LB} &= C^B_{\ell p}\left(\frac{p'_B d}{2V}\right) + C^B_{\ell\Delta}\,\Delta_B \\
C_M &= C_{m\alpha}\,\alpha + C_{mq}\left(\frac{qd}{2V}\right) + C_{n\alpha p}\,\beta\left(\frac{p'_B d}{2V}\right) + C_{m\delta}\,\delta_y \\
C_N &= C_{m\alpha}\,\beta + C_{mq}\left(\frac{rd}{2V}\right) + C_{n\alpha p}\,\alpha\left(\frac{p'_B d}{2V}\right) + C_{m\delta}\,\delta_z
\end{aligned}\right\}\text{Aerodynamic Coefficients}$$

This completes the definitions of the details of a seven degree-of-freedom model of a controlled spinning projectile. All of this detail can be embodied in a single software module which can be represented by the block diagram shown in Figure 3-1. This can be done by partitioning the model into the following parts:

(1) Acceleration due to gravity
(2) Non-dimensional terms
(3) Atmosphere
(4) Aerodynamic forces and moments
(5) Equations of motion
(6) Coordinate transformation and intergration
(7) Direction cosines

This realization is described in the appendix as the realization of Module Number 69060 "Controlled Spinning Projectile". The appendix also describes the realization of Module Number 69220 "7-DOF Projectile Including Seeker and Canard Deflection".

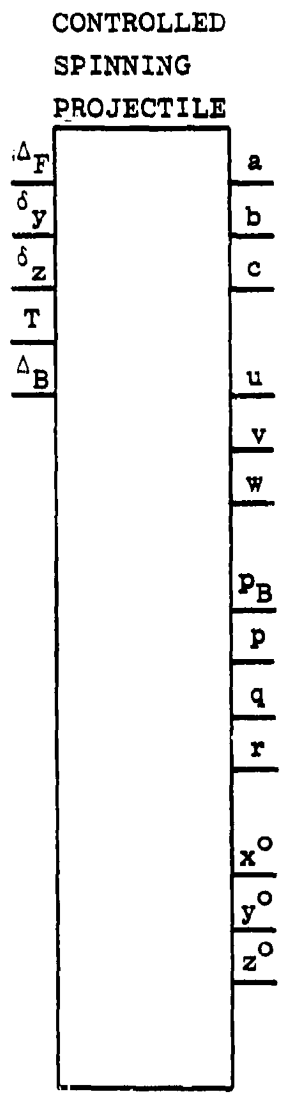


Figure 3-1. Block Diagram Representation of a Seven Degree-of-Freedom Model of a Controlled Spinning Projectile.

SECTION 4

THE FIVE-DEGREE OF FREEDOM MODEL

There are at least two ways of simplifying a mathematical model. One way is by an arbitrary introduction of simplifying approximations. Another is by a careful application of perturbation theory. Both ways will be developed here.

The first way (arbitrary introduction of simplifying assumptions) can be approached as follows:

(1) Assume time histories or constant magnitudes for u and for p_B' and ignore the state equations for $\dot{u}$ and $\dot{p}_B'$.

(2) Evaluate the remaining equations using the assumed magnitudes for u and p_B'.

(3) Discard terms whose magnitudes are judged small enough to be ignorable.

This technique can be pushed so as to obtain linear equations, but it is not necessary to carry it that far.

The equations for the five degree-of-freedom model can be obtained by deleting the appropriate equations from (3-1), (3-2), (3-3) and (3-4) which define the seven degree-of-freedom model. Thus

$$\left.\begin{aligned} b &= Y/m \\ c &= Z/m \end{aligned}\right\} \quad \text{Accelerometer Equations} \qquad (4\text{-}1)$$

$$\left.\begin{aligned} \dot{v} &= -ur + b + g_y \\ \dot{w} &= uq + c + g_z \end{aligned}\right\} \quad \text{Force Equilibrium Equations} \qquad (4\text{-}2)$$

$$\left.\begin{aligned} h_1 &= A_B p_B' + A_F p_F \\ h_2 &= Bq \\ h_3 &= Br \end{aligned}\right\}\quad \text{Angular Momentum Equations} \qquad (4\text{-}3)$$

$$\left.\begin{aligned} \dot{p}_F &= (L+T)/A_F \\ \dot{q} &= (M-h_1 r)/B \\ \dot{r} &= (N+h_1 q)/B \end{aligned}\right\}\quad \text{Moment Equilibrium Equations} \qquad (4\text{-}4)$$

Equations (4-1) through (4-4) are not linear. However, they can be made to approach linearity by the following approximations:

$$\left.\begin{aligned} h_3 p_F &= Brp_F \approx 0 \\ h_2 p_F &= Bqp_F \approx 0 \\ h_1 r &= (A_B p_B + A_F p_F)r \approx (A_B p_B')r \\ h_1 q &= (A_B p_B + A_F p_F)q \approx (A_B p_B')q \end{aligned}\right\} \qquad (4\text{-}5)$$

Substituting (4-5) in equations (4-1) through (4-4) gives:

$$\left.\begin{aligned} b &= Y/m \\ c &= Z/m \\ \dot{v} &= -ur + b + g_y \\ \dot{w} &= \ \ uq + c + g_z \\ \dot{p}_F &= (L+T)/A_F \\ \dot{q} &= (M-A_B p_B' r)/B \\ \dot{r} &= (N+A_B p_B' q)/B \end{aligned}\right\} \qquad (4\text{-}6)$$

It remains to treat the force and moment terms (Y, Z, L, M, N) in a consistent manner, Note that the terms rp_F and qp_F have been dropped because they are second order quantities. The variables v, w, p_F, q, and r are expected to be small, so their products are second order quantities. The product terms have not been dropped because $p_F \neq 0$. The magnitude of p_F must be evaluated and the consequences of its being non-zero must be included.

The required force and moment equations can be selected from equations (3-15), (3-16) and (3-17).

$$\left.\begin{aligned}
Y &= (\rho V^2 S/2) C_y \\
Z &= (\rho V^2 S/2) C_z \\
L &= (\rho V^2 Sd/2) C_{LF} \\
M &= (\rho V^2 Sd/2) C_M \\
N &= (\rho V^2 Sd/2) C_N \\
C_y &= -C_{z\alpha}\beta + C_{z\delta}\delta_z \\
C_z &= C_{z\alpha}\alpha - C_{z\delta}\delta_y \\
C_{L_F} &= C^F_{\ell p}\left(\frac{p_F d}{2V}\right) + C^F_{\ell\Delta}\Delta_F + C^F_{\ell\beta\delta}\beta\delta_y + C^F_{\ell\beta\delta}\alpha\delta_z \\
C_M &= C_{m\alpha}\alpha + C_{mq}\left(\frac{qd}{2V}\right) + C_{n\alpha p}\beta\left(\frac{p_B' d}{2V}\right) + C_{n\delta}\delta_y \\
C_N &= -C_{m\alpha}\beta + C_{mq}\left(\frac{rd}{2V}\right) + C_{n\alpha p}\alpha\left(\frac{p_B' d}{2V}\right) + C_{m\delta}\delta_z
\end{aligned}\right\} \quad (4\text{-}7)$$

Equations (4-7) can be used to express the forces and moments in the following form (see equations 3-21).

$$\left.\begin{aligned}
Y &= \frac{\partial Y}{\partial v}(v + v_w) + \frac{\partial Y}{\partial \delta}\delta_z \\
Z &= \frac{\partial Z}{\partial w}(w + w_w) + \frac{\partial Z}{\partial \delta}\delta_y \\
L &= \frac{\partial L}{\partial p_F} p_F + \frac{\partial L}{\partial \Delta_F}\Delta_F \\
M &= \frac{\partial M}{\partial w}(w + w_w) + \frac{\partial M}{\partial q} q + \frac{\partial M}{\partial v}(v + v_w) + \frac{\partial M}{\partial \delta}\delta_y \\
N &= \frac{\partial N}{\partial v}(v + v_w) + \frac{\partial N}{\partial r} r + \frac{\partial N}{\partial w}(w + w_w) + \frac{\partial N}{\partial \delta}\delta_z
\end{aligned}\right\} \qquad (4\text{-}8)$$

Of course, these partial derivatives are not non-dimensional.

Equations (4-8) have been obtained by neglecting the product terms $\beta\delta_y$ and $\alpha\delta_z$ and by employing the following approximations (see equations (3-21).

$$\left.\begin{aligned}
\beta &\approx \frac{v + v_w}{V} \\
\alpha &\approx \frac{w + w_w}{V}
\end{aligned}\right\} \qquad (4\text{-}9)$$

The expressions for evaluating the partial derivatives are:

$$\left.\begin{aligned}
\frac{\partial Z}{\partial w} &= \frac{\partial Y}{\partial v} = -\frac{\rho V S}{2} C_{z\alpha} \\
\frac{\partial Z}{\partial \delta_y} &= -\frac{\partial Y}{\partial \delta_z} = -\frac{\rho V^2 S}{2} C_{z\delta} \\
\frac{\partial L}{\partial p_F} &= \frac{\rho V S d^2}{4} C^F_{\ell p} \\
\frac{\partial L}{\partial \Delta_F} &= \frac{\rho V^2 S d}{2} C^F_{\ell \Delta} \\
\frac{\partial M}{\partial w} &= -\frac{\partial N}{\partial v} = \frac{\rho V S d}{2} C_{m\alpha} \\
\frac{\partial M}{\partial q} &= \frac{\partial N}{\partial r} = \frac{\rho V S d^2}{4} C_{mq} \\
\frac{\partial M}{\partial v} &= \frac{\partial N}{\partial w} = \frac{\rho S d^2 p_B'}{4} C_{n\alpha p} \\
\frac{\partial M}{\partial \delta} &= \frac{\partial N}{\partial \delta} = \frac{\rho V^2 S d}{2} C_{m\delta}
\end{aligned}\right\} \quad (4\text{-}10)$$

Substituting (4-8) into (4-6) gives

$$\left.\begin{aligned}
b &= [(\partial Y/\partial v)(v + v_w) + (\partial Y/\partial \delta)\delta_z]/m \\
c &= [(\partial Z/\partial w)(w + w_w) + (\partial Z/\partial \delta)\delta_y]/m \\
\dot{v} &= -ur + b + g_y \\
\dot{w} &= uq + c + g_z \\
\dot{p}_F &= [(\partial L/\partial p_F)p_F + (\partial L/\partial \Delta_F)\Delta_F + T]/A_F \\
\dot{q} &= [(\partial M/\partial w)(w+w_w)+(\partial M/\partial q)q+(\partial M/\partial v)(v+v_w) \\
&\qquad + (\partial M/\partial \delta)\delta_y - A_B p_B' r]/B \\
\dot{r} &= [(\partial N/\partial v)(v+v_w)+(\partial N/\partial r)r+(\partial N/\partial w)(w+w_w) \\
&\qquad + (\partial N/\partial \delta)\delta_z + A_B p_B' q]/B
\end{aligned}\right\} \quad (4\text{-}11)$$

$$
\left.
\begin{aligned}
\dot{v} &= (-u)\, r + \frac{1}{m}\left(\frac{\partial Y}{\partial v}\right)(v + v_w) + \frac{1}{m}\left(\frac{\partial Y}{\partial \delta}\right)\delta_z + g_y \\
\dot{w} &= (u)\, q + \frac{1}{m}\left(\frac{\partial Z}{\partial w}\right)(w + w_w) + \frac{1}{m}\left(\frac{\partial Z}{\partial \delta}\right)\delta_y + g_z \\
\dot{p}_F &= \left(\frac{1}{A_F}\,\frac{\partial L}{\partial p_F}\right) p_F + \left(\frac{1}{A_F}\,\frac{\partial L}{\partial \Delta_F}\right)\Delta_F + \frac{T}{A_F} \\
\dot{q} &= \left(\frac{1}{B}\,\frac{\partial M}{\partial w}\right)(w+w_w) + \left(\frac{1}{B}\,\frac{\partial M}{\partial q}\right) q + \left(\frac{1}{B}\,\frac{\partial M}{\partial v}\right)(v+v_w) \\
&\qquad + \left(\frac{1}{B}\,\frac{\partial M}{\partial \delta}\right)\delta_y - \frac{A_B p_B}{B}\, r \\
\dot{r} &= \left(\frac{1}{B}\,\frac{\partial N}{\partial v}\right)(v+v_w) + \left(\frac{1}{B}\,\frac{\partial N}{\partial r}\right) r + \left(\frac{1}{B}\,\frac{\partial N}{\partial w}\right)(w+w_w) \\
&\qquad + \left(\frac{1}{B}\,\frac{\partial N}{\partial \delta}\right)\delta_z + \frac{A_B p_B}{B}\, q
\end{aligned}
\right\} \tag{4-12}
$$

We introduce some more notation

$$
\left.
\begin{aligned}
V_v &= \frac{1}{m}\,\frac{\partial Y}{\partial v} = \frac{1}{m}\,\frac{\partial Z}{\partial w} = W_w = -\frac{\rho V S}{2m}\, C_{z\alpha} \\
-V_r &= W_q = u \\
-V_{\delta_z} &= -\frac{1}{m}\,\frac{\partial Y}{\partial \delta_z} = \frac{1}{m}\,\frac{\partial Z}{\partial \delta_y} = W_{\delta_y} = -\frac{\rho V^2 S}{2m}\, C_{z\delta} \\
P_p &= \frac{1}{A_F}\,\frac{\partial L}{\partial p_F} \\
P_\Delta &= \frac{1}{A_F}\,\frac{\partial L}{\partial \Delta_F} \\
P_T &= \frac{1}{A_F} \\
Q_v &= \frac{1}{B}\,\frac{\partial M}{\partial v} = \frac{1}{B}\,\frac{\partial N}{\partial w} = R_w = \frac{\rho S d^2 p'_B}{4B}\, C_{n\alpha p} \\
Q_w &= \frac{1}{B}\,\frac{\partial M}{\partial w} = -\frac{1}{B}\,\frac{\partial N}{\partial v} = -R_v = \frac{\rho V S d}{2B}\, C_{m\alpha} \\
Q_q &= \frac{1}{B}\,\frac{\partial M}{\partial q} = \frac{1}{B}\,\frac{\partial N}{\partial r} = R_r = \frac{\rho V S d^2}{4B}\, C_{mq} \\
-Q_r &= +R_q = \frac{A_B p'_B}{B} \\
Q_\delta &= \frac{1}{B}\,\frac{\partial M}{\partial \delta} = +\frac{1}{B}\,\frac{\partial N}{\partial \delta} = +R_\delta = \frac{\rho V^2 S d}{2B}\, C_{m\delta}
\end{aligned}
\right\} \tag{4-13}
$$

The equation for $\dot{p}_F$ is quite independent of the other equations. It will be convenient to write our equations in the following form

$$\dot{p}_F = P_p\, p_F + P_\Delta\, \Delta_F + P_T\, T \tag{4-14}$$

$$\begin{bmatrix} \dot{v} \\ \dot{w} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} V_v & 0 & 0 & V_r \\ 0 & W_w & W_q & 0 \\ Q_v & Q_w & Q_q & Q_r \\ R_v & R_w & R_q & R_r \end{bmatrix} \begin{bmatrix} v \\ w \\ q \\ r \end{bmatrix} + \begin{bmatrix} 0 & V_{\delta_z} & 1 & 0 & V_v & 0 \\ W_{\delta_y} & 0 & 0 & 1 & 0 & W_w \\ Q_\delta & 0 & 0 & 0 & Q_v & Q_w \\ 0 & R_\delta & 0 & 0 & R_v & R_w \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_z \\ g_y \\ g_z \\ v_w \\ w_w \end{bmatrix} \tag{4-15}$$

$$\begin{bmatrix} b \\ c \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} V_v & 0 & 0 & 0 \\ 0 & W_w & 0 & 0 \\ 0 & 1/V & 0 & 0 \\ 1/V & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \\ q \\ r \end{bmatrix} + \begin{bmatrix} 0 & V_{\delta_z} & 0 & 0 & V_v & 0 \\ W_{\delta_y} & 0 & 0 & 0 & 0 & W_w \\ 0 & 0 & 0 & 0 & 0 & 1/V \\ 0 & 0 & 0 & 0 & 1/V & 0 \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_z \\ g_y \\ g_z \\ v_w \\ w_w \end{bmatrix} \tag{4-16}$$

The five degree-of-freedom model can be represented by a block diagram such as that shown in Figure 4-1.

Equations (4-14) (4-15) and (4-16) are written with respect to the aeroballistic coordinate frame. The control deflections, δ_y and δ_z, must be generated in this frame or transformed from the fin frame. However, the gravity components and the wind components must be generated in inertial space and transformed onto the unprimed frame.

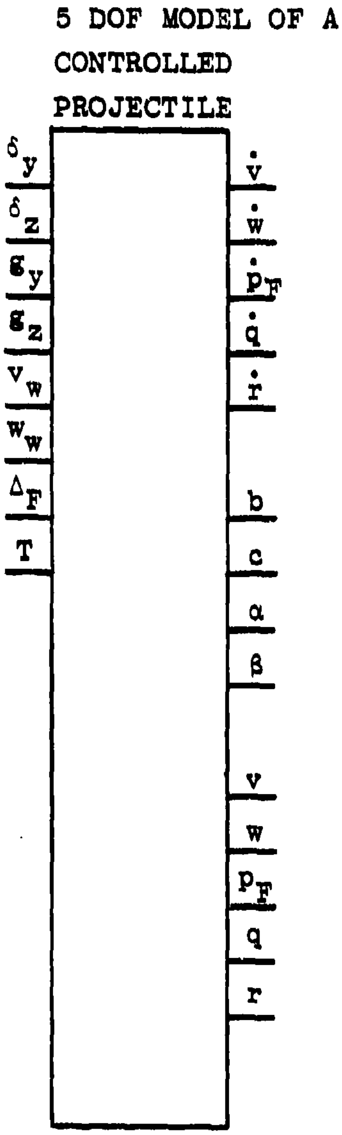


Figure 4-1. Block Diagram Representation of a Five Degree-of-Freedom Model of a Controlled Projectile.

The details of the five degree-of-freedom model can be realized most easily by realizing a four degree-of-freedom model first. The next section (section 5) should be read at this point and forms the basis for the discussion which follows.

The realization of a five degree-of-freedom time-variant model of a controlled projectile is shown in Figure 4-2. This model is a combination of the modules which comprise the four degree-of-freedom model plus additional modules to simulate the moment equilibrium about the x-axis. The spare element in the time-variant element generator can be used to generate the magnitude of p_B. Also, advantage can be taken of the fact that $W_q = u$.

A five degree-of-freedom time-invariant model can be realized as shown in Figure 4-3.

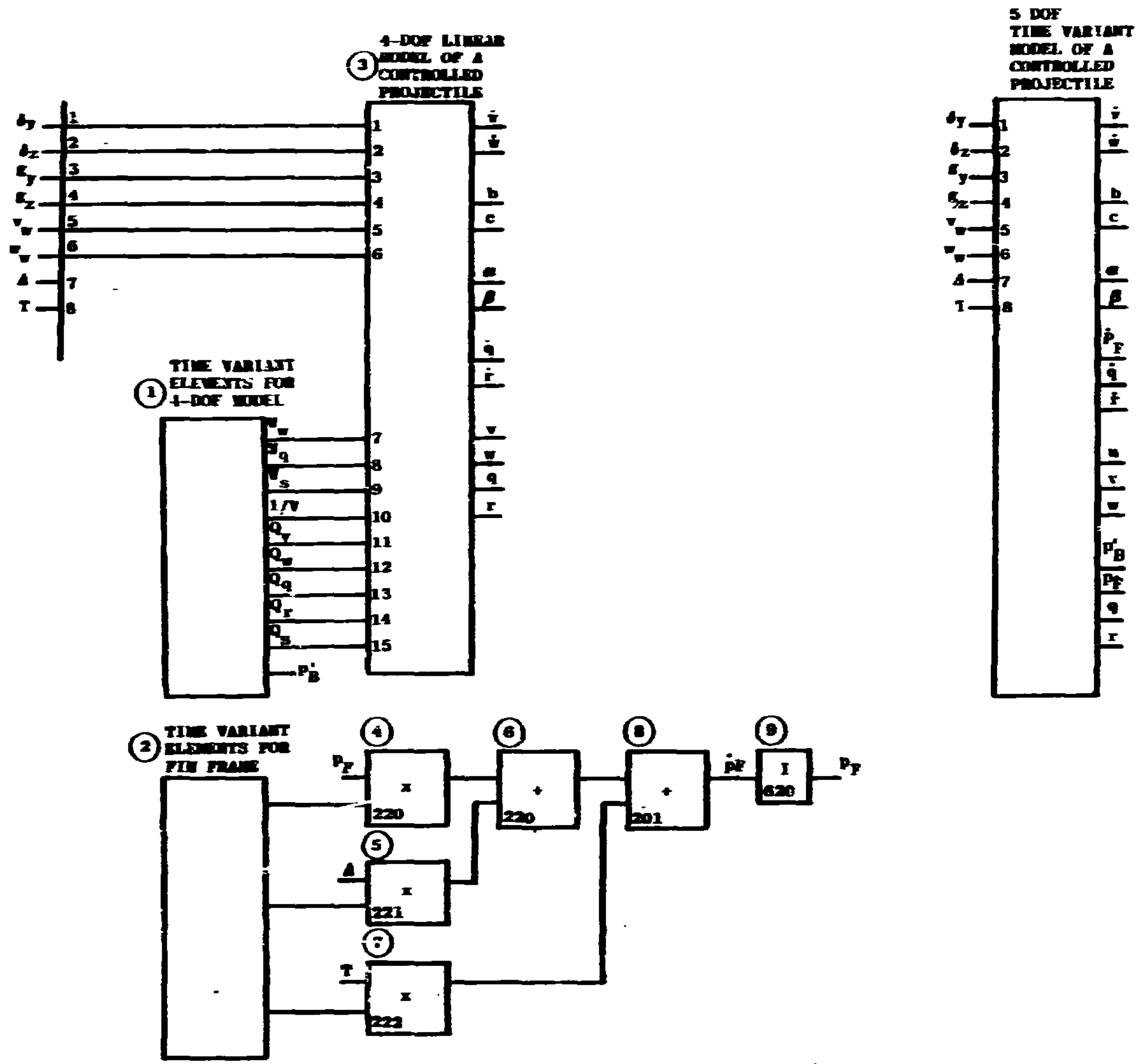


Figure 4-2. Realization of a 5-DOF Time-Variant Model of a Controlled Projectile.

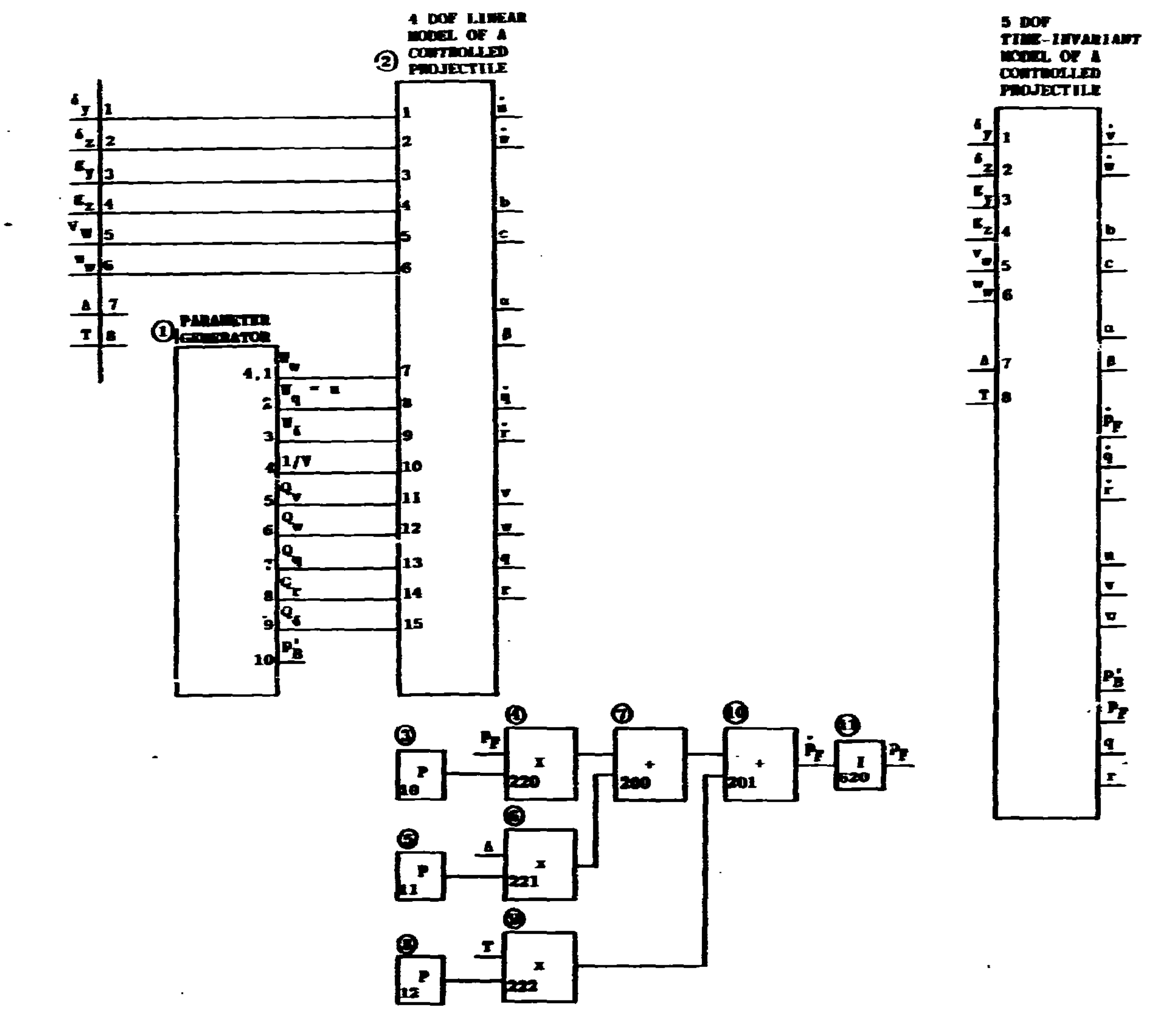


Figure 4-3. Realization of a 5-DOF Time-Invariant Model or a Controlled Projectile.

SECTION 5

THE FOUR DEGREE-OF-FREEDOM MODEL

The four degree-of-freedom model can be obtained readily from the five degree-of-freedom model by setting $\dot{p}_F = p_F = 0$. Then

$$\left.\begin{aligned} b &= Y/m \\ c &= Z/m \end{aligned}\right\} \quad \text{Accelerometer Equations} \qquad (5\text{-}1)$$

$$\left.\begin{aligned} \dot{v} &= -ur + b + g_y \\ \dot{w} &= uq + c + g_z \end{aligned}\right\} \quad \text{Force Equilibrium Equations} \qquad (5\text{-}2)$$

$$\left.\begin{aligned} h_1 &= A_B p_B' \\ h_2 &= Bq \\ h_3 &= Br \end{aligned}\right\} \quad \text{Angular Momentum Equations} \qquad (5\text{-}3)$$

$$\left.\begin{aligned} \dot{q} &= (M - h_1 r)/B \\ \dot{r} &= (N + h_1 q)/B \end{aligned}\right\} \quad \text{Moment Equilibrium} \qquad (5\text{-}4)$$

These equations will lead to equations which are equivalent to equations (4-15), (4-16) and a subset of (4-13). These equations are rewritten here so as to have a compact summary.

$$\left.\begin{aligned} V_v &= W_w = -\frac{\rho VS}{2m} C_{z\alpha} \\ -V_r &= W_q = u \\ -W_{\delta_z} &= W_{\delta_y} = -\frac{\rho V^2 S}{2m} C_{z\delta} \\ Q_v &= R_w = \frac{\rho S d^2 p_B'}{4B} C_{n\alpha p} \\ Q_w &= -R_v = \frac{\rho VSd}{2B} C_{m\alpha} \end{aligned}\right\}$$

$$
\left.
\begin{aligned}
Q_q &= R_r = \frac{\rho V S d^2}{4B} C_{mq} \\
-Q_r &= + R_q = + \frac{A_B p_B}{B} \\
Q_\delta &= + R_\delta = \frac{\rho V^2 S d}{2B} C_{m\delta}
\end{aligned}
\right\} \qquad (5\text{-}5)
$$

$$
\begin{bmatrix} \dot{v} \\ \dot{w} \\ \dot{q} \\ \dot{r} \end{bmatrix}
=
\begin{bmatrix} V_v & 0 & 0 & V_r \\ 0 & W_w & W_q & 0 \\ Q_v & Q_w & Q_q & Q_r \\ R_v & R_w & R_q & R_r \end{bmatrix}
\begin{bmatrix} v \\ w \\ q \\ r \end{bmatrix}
+
\begin{bmatrix} 0 & V_{\delta_z} & 1 & 0 & V_v & 0 \\ W_{\delta_y} & 0 & 0 & 1 & 0 & W_w \\ Q_\delta & 0 & 0 & 0 & Q_v & Q_w \\ 0 & R_\delta & 0 & 0 & R_v & R_w \end{bmatrix}
\begin{bmatrix} \delta_y \\ \delta_z \\ g_y \\ g_z \\ v_w \\ w_w \end{bmatrix}
\qquad (5\text{-}6)
$$

$$
\begin{bmatrix} b \\ c \\ \alpha \\ \beta \end{bmatrix}
=
\begin{bmatrix} V_v & 0 & 0 & 0 \\ 0 & W_w & 0 & 0 \\ 0 & 1/V & 0 & 0 \\ 1/V & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} v \\ w \\ q \\ r \end{bmatrix}
+
\begin{bmatrix} 0 & V_{\delta_z} & 0 & 0 & V_v & 0 \\ W_{\delta_y} & 0 & 0 & 0 & 0 & W_w \\ 0 & 0 & 0 & 0 & 0 & 1/V \\ 0 & 0 & 0 & 0 & 1/V & 0 \end{bmatrix}
\begin{bmatrix} \delta_y \\ \delta_z \\ g_y \\ g_z \\ v_w \\ w_w \end{bmatrix}
\qquad (5\text{-}7)
$$

Equations (5-6) and (5-7) are linear, but the matrix elements may be time variant.

In order to accommodate the time-variant nature of the four degree-of-freedom model it will be organized as shown in Figures 5-1, 5-2 and 5-3. The time-varying coefficients are the magnitudes given in in equation 4-5. These coefficients can be determined in a 7 DOF simulation and then expressed as polynominals. Quadratic polynomials will be reasonable at this time.

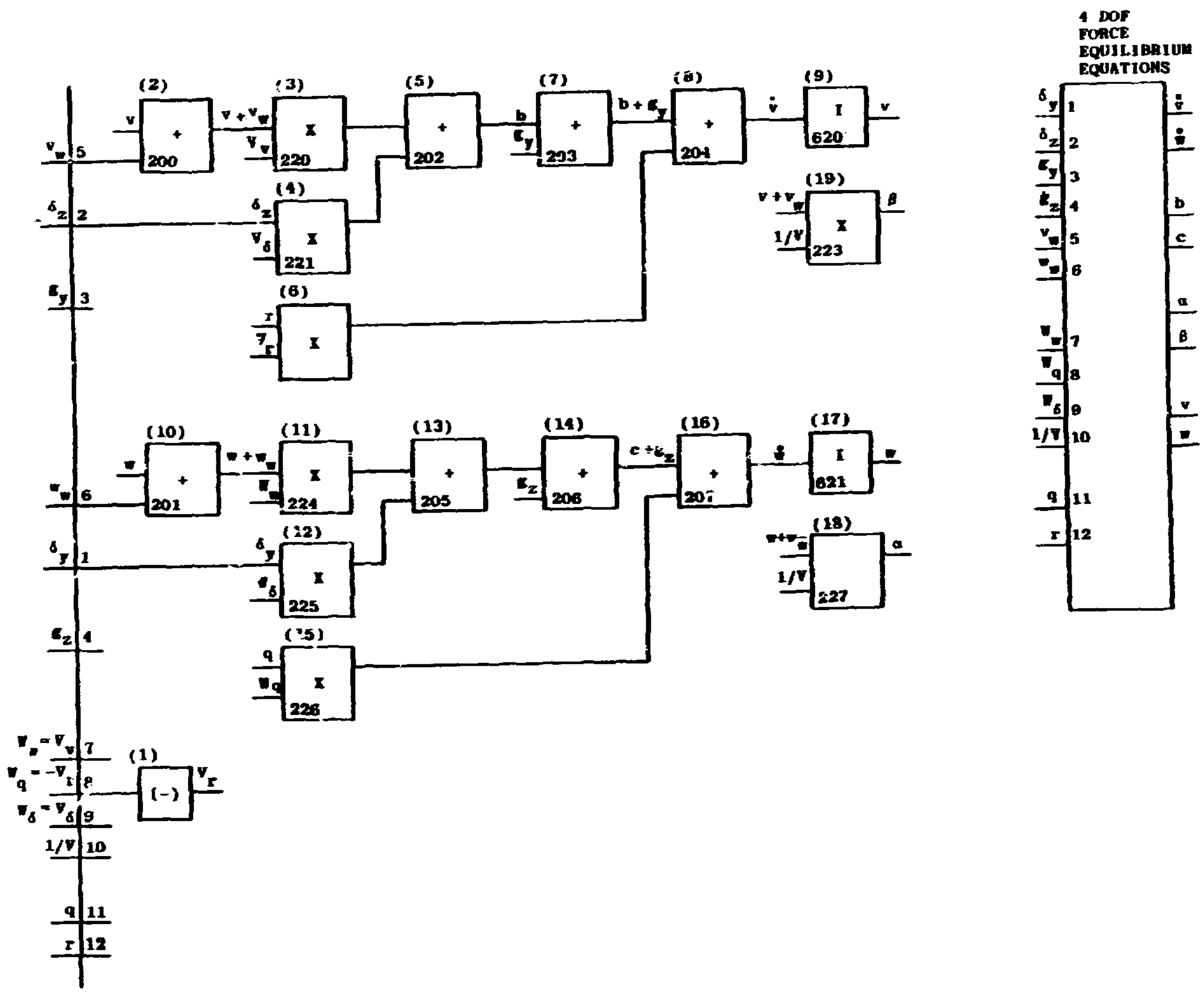


Figure 5-1. Block Diagram of 4-DOF Force Equilibrium Equations.

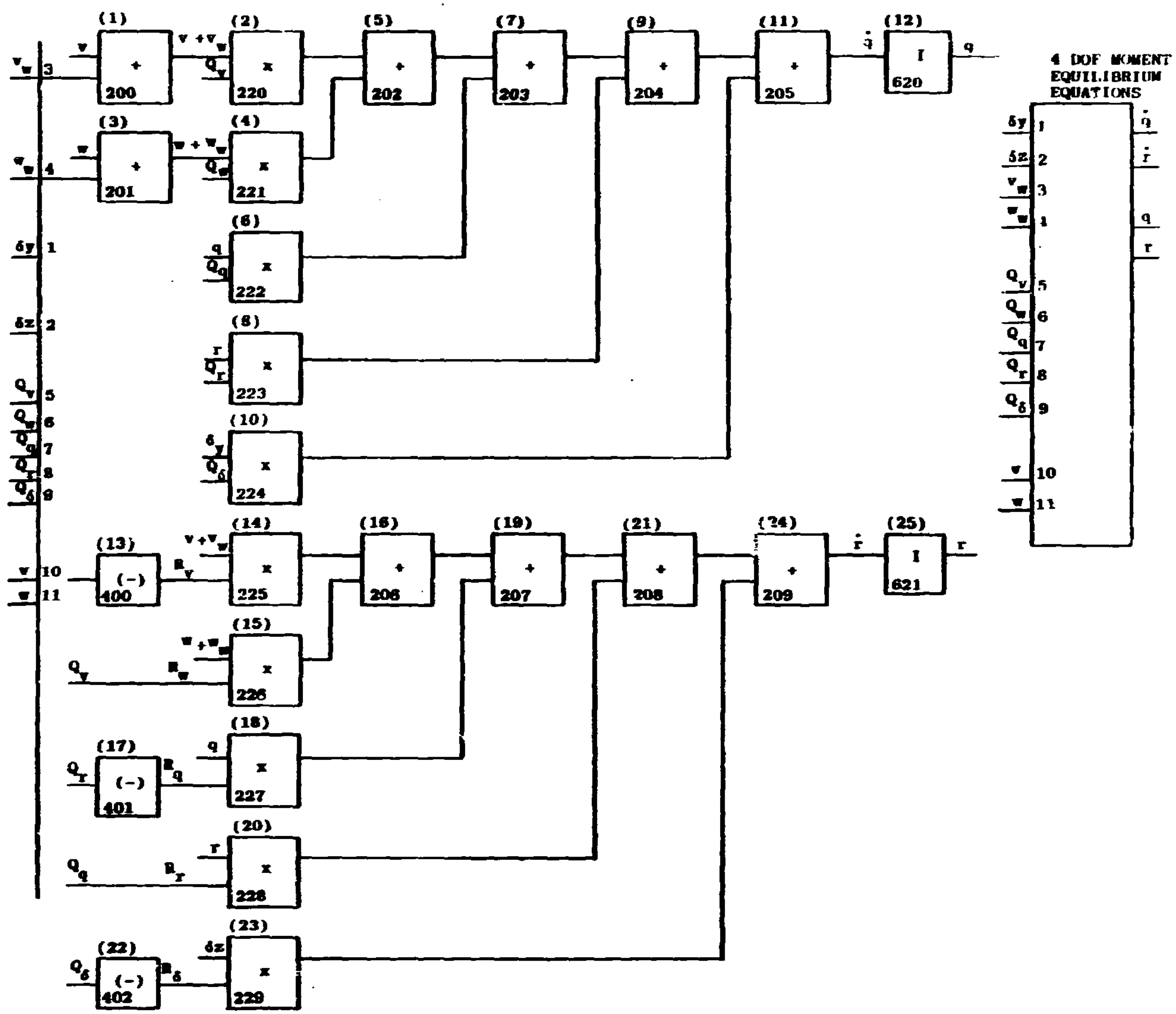


Figure 5-2. 4-DOF Moment Equilibrium Equations.

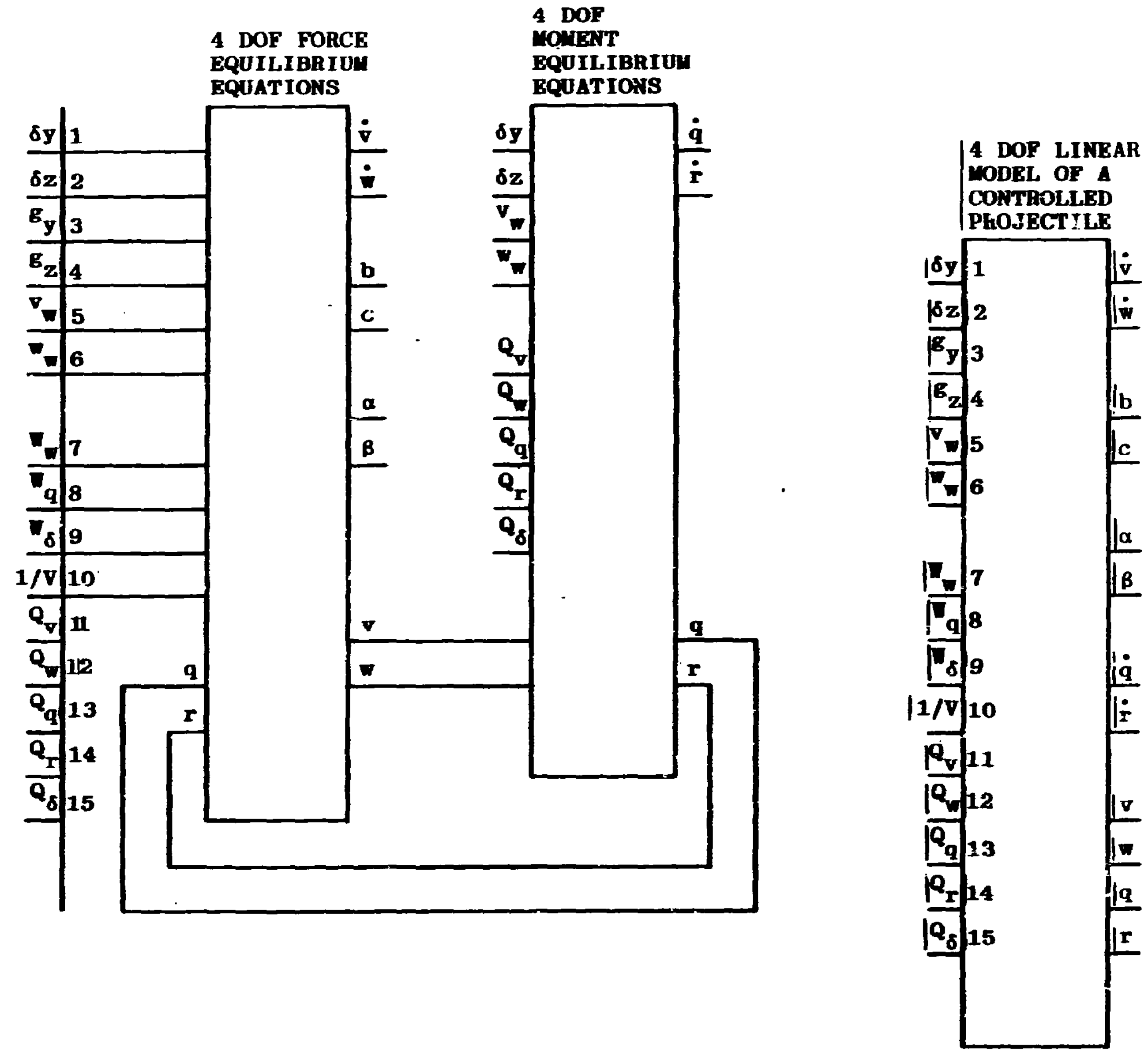


Figure 5-3. Block Diagram of a 4-DOF Model of a Controlled Projectile.

Quadratic polynomials can be generated by module number 8130. Ten of these can be combined in a single module as is shown in Figure 5-4.

The realizations of time-variant and time-invariant 4 degree-of-freedom models are described and compared in Figure 5-5.

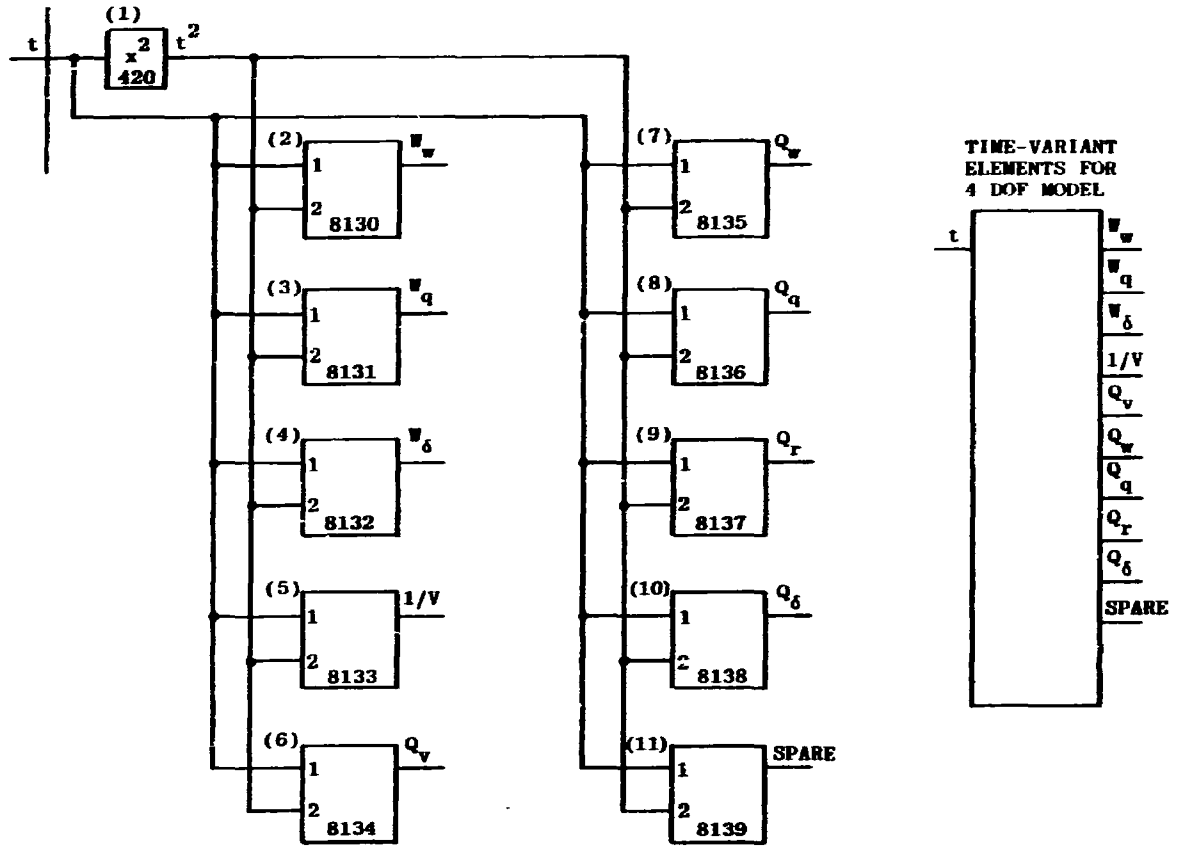
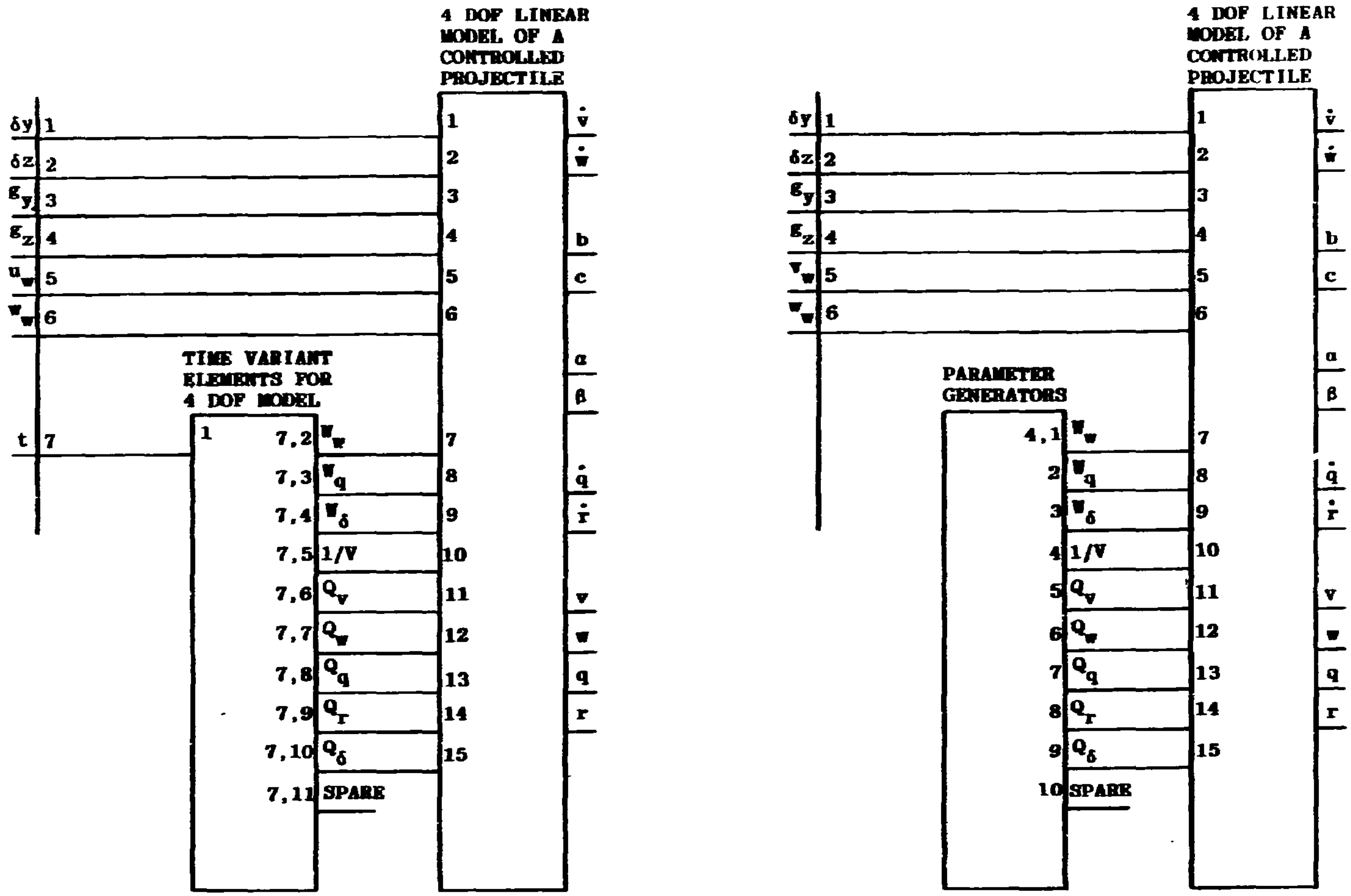


Figure 5-4. Block Diagram of Time Variant Elements for 4-DOF Model.



(a) TIME-VARIANT 4 DOF LINEAR MODEL OF A CONTROLLED PROJECTILE

(b) TIME-INVARIANT 4 DOF LINEAR MODEL OF A CONTROLLED PROJECTILE

Figure 5-5. Comparison of a Time-Variant and a Time-Invariant 4-DOF Linear Model of a Controlled Projectile.

SECTION 6

SYSTEM DESCRIPTIONS EMPLOYING EXACT EXPRESSIONS FOR THE COORDINATE TRANSFORMATIONS

The approximate five and four degree-of-freedom models can be employed in conjunction with exact or approximate expressions for the coordinate transformations. The approximate expressions are more eonomical of computing power and permit linearization and, therefore they will be used whenever possible. However, the exact expressions will be useful for error control.

The complete description of the projectile, which parallels that presented for the seven degree-of-freedom model (see Section 3) is represented in Figure 6-1.

The target can be represented by the block diagram shown in Figure 6-2.

The relative motion of the projectile and the target is described by the following equations:

$$\left.\begin{aligned} \Delta u^o &= u_T^o - u_P^o \\ \Delta v^o &= v_T^o - v_P^o \\ \Delta w^o &= w_T^o - w_P^o \end{aligned}\right\} \qquad (6\text{-}1)$$

$$\left.\begin{aligned} \Delta x^o &= x_T^o - x_P^o \\ \Delta y^o &= y_T^o - y_P^o \\ \Delta z^o &= z_T^o - z_P^o \end{aligned}\right\} \qquad (6\text{-}2)$$

$$R^2 = \overline{\Delta x^o}^{\,2} + \overline{\Delta y^o}^{\,2} + \overline{\Delta z^o}^{\,2} \qquad (6\text{-}3)$$

$$R\dot{R} = \Delta x^{o} \frac{d}{dt} \Delta x^{o} + \Delta y^{o} \frac{d}{dt} \Delta y^{o} + \Delta z^{o} \frac{d}{dt} \Delta z^{o} \tag{6-4}$$

$$\left.\begin{aligned} \Delta u^{o} &= \frac{d}{dt} \Delta x^{o} \\ \Delta v^{o} &= \frac{d}{dt} \Delta y^{o} \\ \Delta w^{o} &= \frac{d}{dt} \Delta z^{o} \end{aligned}\right\} \tag{6-5}$$

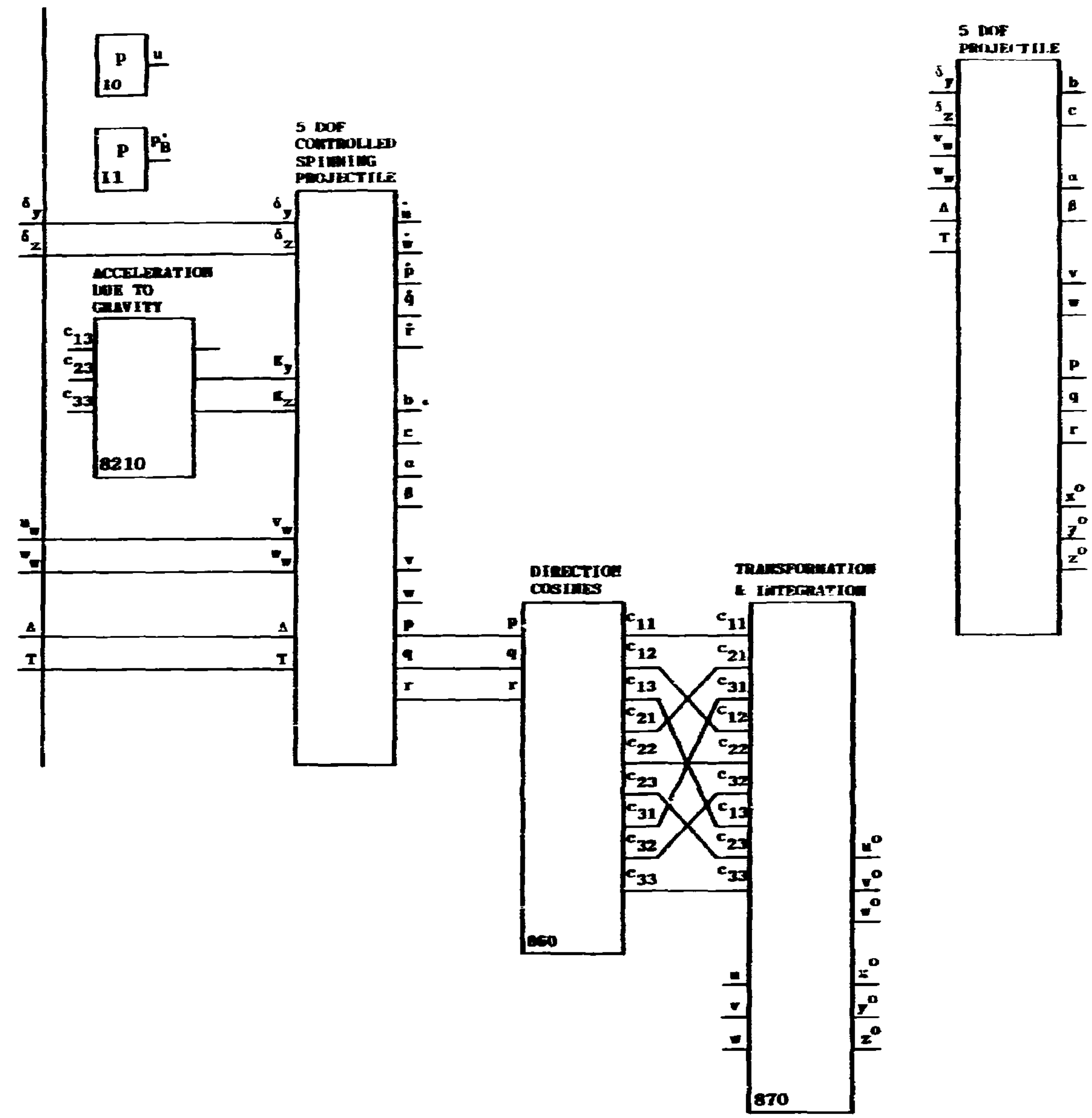


Figure 6-1. 5 DOF Projectile with Exact Coordinate Transformation.

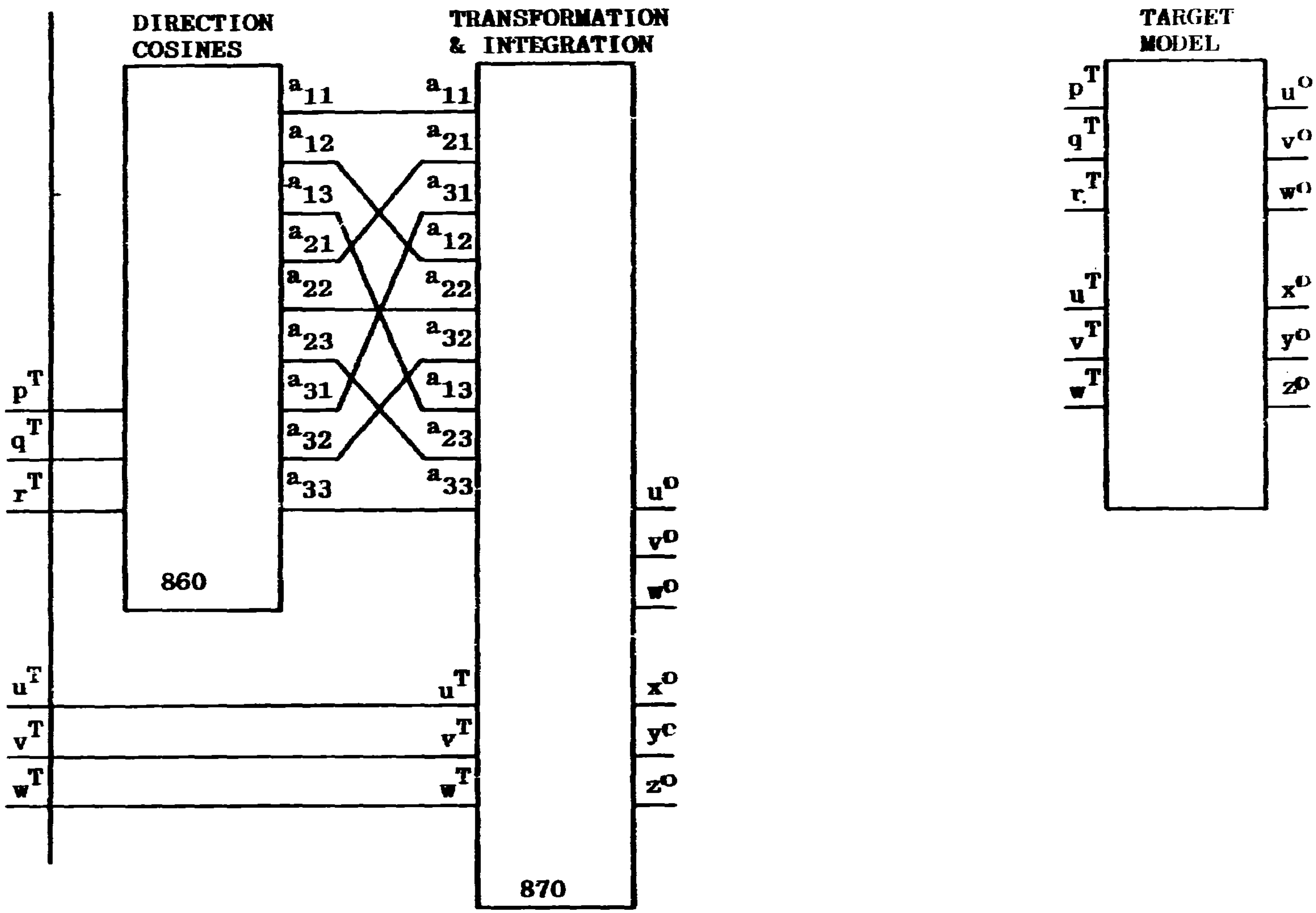


Figure 6-2. Target Model with Exact Coordinate Transformation.

Then,

$$\left.\begin{aligned} R\dot{R} &= \Delta x^o\ \Delta u^o + \Delta y^o\ \Delta v^o + \Delta z^o\ \Delta w^o \\ \dot{R} &= \frac{\Delta x^o}{R}\Delta u^o + \frac{\Delta y^o}{R}\ \Delta v^o + \frac{\Delta z^o}{R}\ \Delta w^o \end{aligned}\right\} \qquad (6\text{-}6)$$

A seeker mounted in the aeroballistic frame (unprimed coordinate frame) will develop boresight-error angles which depend on the position of the target relative to the projectile, resolved onto the unprimed coordinate frame. Thus,

$$\overline{\Delta X} = \mathbf{C}\ \overline{\Delta X^o} \quad \text{(see Appendix A)} \qquad (6\text{-}7)$$

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \Delta x^o \\ \Delta y^o \\ \Delta z^o \end{bmatrix} \qquad (6\text{-}8)$$

The boresight-error angles will be

$$\left.\begin{aligned} \lambda_y &= \tan^{-1}\left(-\frac{\Delta z}{\Delta x}\right) \\ \lambda_z &= \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) \end{aligned}\right\} \qquad (6\text{-}9)$$

The boresight-error angles measure the relative position of the target and projectile, but they also include the effect of attitude change of the projectile. This is not obvious from the exact equations; but it will be obvious from the approximate equations which are developed in the following section.

The relative motion of the target and the projectile can be represented by the block shown in Figure 6-3. This block also

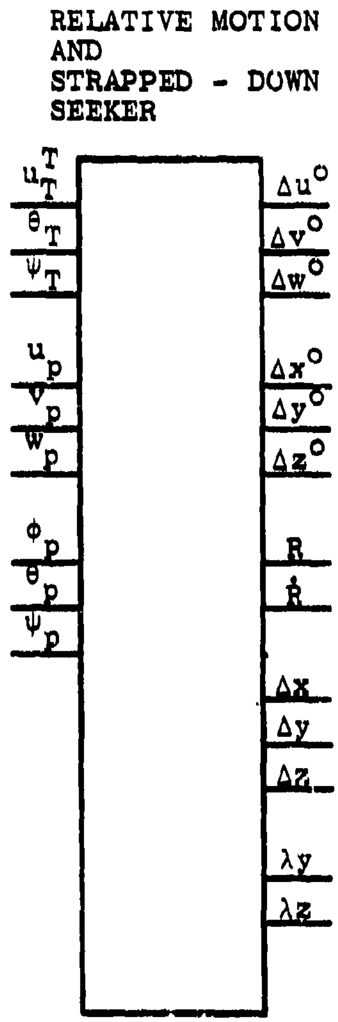


Figure 6-3. Block Representing the Relative Motion of Target and Projectile and the geometric processing performed by a Strapped-Down Seeker.

represents the geometric processing done by a strapped down seeker and generates the boresight-error angles, λ_y and λ_z. These functions are valid for the 7-DOF, 5-DOF, and 4-DOF models.

SECTION 7

SYSTEM DESCRIPTIONS EMPLOYING APPROXIMATE EXPRESSIONS FOR THE COORDINATE TRANSFORMATIONS

System descriptions employing approximate expressions for the coordinate transformations permit linearization in a manner comparable to that which has been introduced for the four and five degree of freedom models. Such treatment permits the linearization of the complete interception problem. The linearized state-space equations for the physical plant are:

$$\left.\begin{aligned} \underline{\dot{x}}(t) &= \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) \\ \underline{y}(t) &= \underline{C}(t)\underline{x}(t) + \underline{D}(t)\underline{u}(t) \end{aligned}\right\} \qquad (7\text{-}1)$$

where

$\underline{x}(t)$ = a state vector

$\underline{y}(t)$ = an output vector

$\underline{u}(t)$ = an input vector

The state vector will include the following terms:

p_F angular velocity of the fin frame (rad/sec)

$\left.\begin{matrix} v_p \\ w_p \end{matrix}\right\}$ translational velocity (ft/sec)

$\left.\begin{matrix} q \\ r \end{matrix}\right\}$ angular velocity (rad/sec)

$$\left.\begin{array}{l}\phi_F \\ \theta \\ \psi\end{array}\right\}\text{ rotations (rad)}$$

$$\left.\begin{array}{l}y \\ z\end{array}\right\}\text{ translational perturbations (ft)}$$

δ control deflection (rad)

The output vector will include the following terms:

$$\left.\begin{array}{l}b \\ c\end{array}\right\}\text{ translational acceleration (ft/sec}^2\text{)}$$

α angle-of-attack (rad)

β side-slip angle (rad)

$$\left.\begin{array}{l}\Delta y \\ \Delta z\end{array}\right\}\text{ relative position components (ft)}$$

$$\left.\begin{array}{l}\lambda_y \\ \lambda_z\end{array}\right\}\text{ look angle components (rad)}$$

The input vector will include the following terms:

$$\left.\begin{array}{l}u_w^o \\ v_w^o \\ w_w^o\end{array}\right\}\text{ wind components (ft/sec)} \qquad \Delta_F \text{ fin cant angle (rad)}$$

$$\left.\begin{array}{l}\ddot{\theta}_T \\ \dot{\psi}_T\end{array}\right\}\text{ target maneuvers (rad/sec)}$$

The approximate expressions developed in this section will employ a coordinate frame which is fixed to the cruciform canards (the x^F, y^F, z^F). This convention makes it possible to include the effects of small perturbations, ϕ, in the roll angle of the canard frame. The perturbation equations for the direction cosines are similar to those which have been developed in equations (2-10).

$$\left.\begin{aligned} \Delta f_{1j} &= f_{2j}\psi - f_{3j}\theta \\ \Delta f_{2j} &= f_{3j}\phi - f_{1j}\psi \\ \Delta f_{3j} &= f_{1j}\theta - f_{2j}\phi \end{aligned}\right\} \qquad (7\text{-}2)$$

where

$$\left.\begin{aligned} \phi &= \phi\,(0) + \int_0^t p_F^F dt \\ \theta &= \theta\,(0) + \int_0^t q_F^F dt \\ \psi &= \psi\,(0) + \int_0^t r_F^F dt \end{aligned}\right\} \qquad (7\text{-}3)$$

The resolution of gravity onto the fin frame is described by:

$$\begin{bmatrix} g_x^F + \Delta g_x^F \\ g_y^F + \Delta g_y^F \\ g_z^F + \Delta g_z^F \end{bmatrix} = \begin{bmatrix} f_{11} + \Delta f_{11} & f_{12} + \Delta f_{12} & f_{13} + \Delta f_{13} \\ f_{21} + \Delta f_{21} & f_{22} + \Delta f_{22} & f_{23} + \Delta f_{23} \\ f_{31} + \Delta f_{31} & f_{32} + \Delta f_{32} & f_{33} + \Delta f_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = g \begin{bmatrix} f_{13} + \Delta f_{13} \\ f_{33} + \Delta f_{23} \\ f_{33} + \Delta f_{33} \end{bmatrix} \qquad (7\text{-}4)$$

The usual practice will be to include the effect of gravity in the reference trajectory. Then it is only necessary to treat Δg in the linearized equations. Thus:

$$
\begin{bmatrix} \Delta g_x^F \\ \Delta g_y^F \\ \Delta g_z^F \end{bmatrix} = \begin{bmatrix} 0 & -f_{33} & f_{23} \\ f_{33} & 0 & -f_{13} \\ -f_{23} & f_{13} & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} g \tag{7-5}
$$

The transformation of the wind components onto the fin frame is given by:

$$
\begin{bmatrix} u_w^F \\ v_w^F \\ w_w^F \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u_w^O \\ v_w^O \\ w_w^O \end{bmatrix} \tag{7-6}
$$

No perturbation terms are included in the transformation because the wind components are considered to be small quantities.

The velocity components, computed in the fin frame, must next be resolved onto the inertial frame. Thus:

$$
\begin{bmatrix} u_P^O \\ v_P^O \\ w_P^O \end{bmatrix} = \begin{bmatrix} f_{11}+f_{21}\psi-f_{31}\theta & f_{21}+f_{31}\phi-f_{11}\psi & f_{31}+f_{11}\theta-f_{21}\phi \\ f_{12}+f_{22}\psi-f_{32}\theta & f_{22}+f_{32}\phi-f_{12}\psi & f_{32}+f_{12}\theta-f_{22}\phi \\ f_{13}+f_{23}\psi-f_{33}\theta & f_{23}+f_{33}\phi-f_{13}\psi & f_{33}+f_{13}\theta-f_{23}\phi \end{bmatrix} \begin{bmatrix} u_P^F \\ v_P^F \\ w_P^F \end{bmatrix} \tag{7-7}
$$

Expanding equation (7-7) and remembering that v_P^F and w_P^F are small perturbations:

$$\begin{bmatrix} u_P^O \\ v_P^O \\ w_P^O \end{bmatrix} = \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix} \begin{bmatrix} u_P^F \\ v_P^F + u_P^F\psi \\ w_P^F - u_P^F\theta \end{bmatrix} \tag{7-8}$$

The target's velocity can be specified so that $v_T^T = w_T^T = 0$. Then the target's velocity can be resolved onto the inertial frame in a similar manner and:

$$\begin{bmatrix} u_T^O \\ v_T^O \\ w_T^O \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} u_T^T \\ u_T^T\psi_T \\ -u_T^T\theta_T \end{bmatrix} \tag{7-9}$$

The position coordinates of the target are:

$$\left.\begin{aligned} x_T^O &= x_T^O(0) + \int_0^t u_T^T\,(a_{11} + a_{21}\psi_T - a_{31}\theta_T)dt \\ y_t^O &= y_T^O(0) + \int_0^t u_T^T\,(a_{12} + a_{22}\psi_T - a_{32}\theta_T)dt \\ z_T^O &= z_T^O(0) + \int_0^t u_T^T\,(a_{13} + a_{23}\psi_T - a_{33}\theta_T)dt \end{aligned}\right\} \tag{7-10}$$

and the coordinates of the projectile are:

$$\left.\begin{aligned} x_P^O &= x_P^O(0) + \int_0^t \left[f_{11}\, u_P^F + f_{21}\,(v_P^F + u_P^F\psi\,) + f_{31}\,(w_P^F - u_P^F\theta\,)\right] dt \\ y_P^O &= y_P^O(0) + \int_0^t \left[f_{12}\, u_P^F + f_{22}\,(v_P^F + u_P^F\psi\,) + f_{32}\,(w_P^F - u_P^F\theta\,)\right] dt \\ z_P^O &= z_P^O(0) + \int_0^t \left[f_{13}\, u_P^F + f_{23}\,(v_P^F + u_P^F\psi\,) + f_{33}\,(w_P^F - u_P^F\theta\,)\right] dt \end{aligned}\right\} \tag{7-11}$$

The relative position components of target and projectile, defined in the inertial frame are:

$$\left.\begin{aligned} \Delta x^o &= x_T^o - x_P^o = \Delta\tilde{x}^o - \Delta\hat{x}^o \\ \Delta y^o &= y_T^o - y_P^o = \Delta\tilde{y}^o - \Delta\hat{y}^o \\ \Delta z^o &= z_T^o - z_P^o = \Delta\tilde{z}^o - \Delta\hat{z}^o \end{aligned}\right\} \tag{7-12}$$

where

$$\left.\begin{aligned} \Delta\tilde{x}^o &= x_T^o(0)-x_P^o(0)+\int_0^t (a_{11}u_T^T-f_{11}u_P^F)\,dt+\int_0^t u_T^T\,(a_{21}\psi_T-a_{31}\theta_T)\,dt \\ \Delta\tilde{y}^o &= y_T^o(0)-y_P^o(0)+\int_0^t (a_{12}u_T^T-f_{12}u_P^F)\,dt+\int_0^t u_T^T\,(a_{22}\psi_T-a_{32}\theta_T)\,dt \\ \Delta\tilde{z}^o &= z_T^o(0)-z_P^o(0)+\int_0^t (a_{13}u_T^T-f_{13}u_P^F)\,dt+\int_0^t u_T^T\,(a_{23}\psi_T-a_{33}\theta_T)\,dt \end{aligned}\right\} \tag{7-13}$$

and

$$\left.\begin{aligned} \Delta\hat{x}^o &= \int_0^t \left[f_{21}\,(v_P^F + u_P^F\psi\,) + f_{31}\,(w_P^F - u_P^F\theta\,)\right] dt \\ \Delta\hat{y}^o &= \int_0^t \left[f_{22}\,(v_P^F + u_P^F\psi\,) + f_{32}\,(w_P^F - u_P^F\theta\,)\right] dt \\ \Delta\hat{z}^o &= \int_0^t \left[f_{23}\,(v_P^F + u_P^F\psi\,) + f_{33}\,(w_P^F - u_P^F\theta\,)\right] dt \end{aligned}\right\} \tag{7-14}$$

It can be seen that the components of $\Delta\tilde{X}^o$, $\Delta\tilde{Y}^o$, $\Delta\tilde{Z}^o$ are functions of time, whereas the components of $\Delta\hat{x}^o$, $\Delta\hat{y}^o$, $\Delta\hat{z}^o$ are functions of the perturbations v_P^F, w_P^F, θ and ψ. These relative displacement components must now be transformed back to the fin frame.

$$
\begin{bmatrix} \Delta x^F \\ \Delta y^F \\ \Delta z^F \end{bmatrix} = \left\{ \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} + \begin{bmatrix} f_{21}\psi - f_{31}\theta & f_{22}\psi - f_{32}\theta & f_{23}\psi - f_{33}\theta \\ f_{31}\phi - f_{11}\psi & f_{32}\phi - f_{12}\psi & f_{33}\phi - f_{13}\psi \\ f_{11}\theta - f_{21}\phi & f_{12}\theta - f_{22}\phi & f_{13}\theta - f_{23}\phi \end{bmatrix} \right\} \begin{bmatrix} \widetilde{\Delta x}^O + \Delta\hat{x}^O \\ \widetilde{\Delta y}^O + \Delta\hat{y}^O \\ \widetilde{\Delta z}^O + \Delta\hat{z}^O \end{bmatrix} \tag{7-15}
$$

The right hand side of equation (7-15) expands into four terms which are treated below, one at a time.

The first term is:

$$
\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} \widetilde{\Delta x}^O \\ \widetilde{\Delta z}^O \\ \widetilde{\Delta z}^O \end{bmatrix} = \begin{bmatrix} \widetilde{\Delta x}^F \\ \widetilde{\Delta y}^F \\ \widetilde{\Delta z}^F \end{bmatrix} \tag{7-16}
$$

The second term is:

$$
\begin{bmatrix} f_{21}\psi - f_{31}\theta & f_{22}\psi - f_{32}\theta & f_{23}\psi - f_{33}\theta \\ f_{31}\phi - f_{11}\psi & f_{32}\phi - f_{12}\psi & f_{33}\phi - f_{13}\psi \\ f_{11}\theta - f_{21}\phi & f_{12}\theta - f_{22}\phi & f_{13}\theta - f_{23}\phi \end{bmatrix} \begin{bmatrix} \widetilde{\Delta x}^O \\ \widetilde{\Delta y}^O \\ \widetilde{\Delta z}^O \end{bmatrix}
$$

$$
= \begin{bmatrix} (f_{21}\widetilde{\Delta x}^O + f_{22}\widetilde{\Delta y}^O + f_{23}\widetilde{\Delta z}^O)\psi - (f_{31}\widetilde{\Delta x}^O + f_{32}\widetilde{\Delta y}^O + f_{33}\widetilde{\Delta z}^O)\theta \\ (f_{31}\widetilde{\Delta x}^O + f_{32}\widetilde{\Delta y}^O + f_{33}\widetilde{\Delta z}^O)\phi - (f_{11}\widetilde{\Delta x}^O + f_{12}\widetilde{\Delta y}^O + f_{13}\widetilde{\Delta z}^O)\psi \\ (f_{11}\widetilde{\Delta x}^O + f_{12}\widetilde{\Delta y}^O + f_{13}\widetilde{\Delta z}^O)\theta - (f_{21}\widetilde{\Delta x}^O + f_{22}\widetilde{\Delta y}^O + f_{23}\widetilde{\Delta z}^O)\phi \end{bmatrix}
$$

$$
= \begin{bmatrix} \widetilde{\Delta y}^F\psi - \widetilde{\Delta z}^F\theta \\ \widetilde{\Delta z}^F\phi - \widetilde{\Delta x}^F\psi \\ \widetilde{\Delta x}^F\theta - \widetilde{\Delta y}^F\phi \end{bmatrix} \tag{7-17}
$$

The third term is:

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} \Delta\tilde{x}^o \\ \Delta\tilde{y}^o \\ \Delta\tilde{z}^o \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} \int_0^t \left[f_{21} \; (v_p^F + u_p^F \psi \;) + f_{31} \; (w_p^F - u_p^F \theta) \right] dt \\ \int_0^t \left[f_{22} \; (v_p^F + u_p^F \psi \;) + f_{32} \; (w_p^F - u_p^F \theta) \right] dt \\ \int_0^t \left[f_{23} \; (v_p^F + u_p^F \psi \;) + f_{33} \; (w_p^F - u_p^F \theta) \right] dt \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^t \left[(f_{11} \; f_{21} + f_{12} \; f_{22} + f_{13} \; f_{23})(v_p^F + u_p^F \; \psi) + (f_{11} \; f_{31} + f_{12} f_{32} + f_{13} f_{33})(w_p^F - u_p^F \theta) \right] dt \\ \int_0^t \left[(f_{21} \; f_{21} + f_{22} \; f_{22} + f_{23} \; f_{23})(v_p^F + u_p^F \; \psi) + (f_{21} \; f_{31} + f_{22} f_{32} + f_{23} f_{33})(w_p^F - u_p^F \theta) \right] dt \\ \int_0^t \left[(f_{31} \; f_{21} + f_{32} \; f_{22} + f_{33} \; f_{23})(v_p^F + u_p^F \; \psi) + (f_{31} \; f_{31} + f_{32} f_{32} + f_{33} f_{33})(w_p^F - u_p^F \theta) \right] dt \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \int (v_p^F + u_p^F \; \psi) \; dt \\ \int (w_p^F - u_p^F \; \theta) \; dt \end{bmatrix} \qquad (7\text{-}18)$$

The fourth term is:

$$\begin{bmatrix} f_{21}\psi - f_{31}\theta & f_{22}\psi - f_{32}\theta & f_{23}\psi - f_{33}\theta \\ f_{31}\phi - f_{11}\psi & f_{32}\phi - f_{12}\psi & f_{33}\phi - f_{13}\psi \\ f_{11}\theta - f_{21}\phi & f_{12}\theta - f_{22}\phi & f_{13}\theta - f_{23}\phi \end{bmatrix} \begin{bmatrix} \Delta\hat{x}^{o} \\ \Delta\hat{y}^{o} \\ \Delta\hat{z}^{o} \end{bmatrix} \tag{7-19}$$

The expansion of this term contains only products of perturbations, all of which can be ignored. Then the expansion of equation (7-18) is:

$$\begin{bmatrix} \Delta x^{F} \\ \Delta y^{F} \\ \Delta z^{F} \end{bmatrix} = \begin{bmatrix} \Delta\tilde{x}^{F} \\ \Delta\tilde{y}^{F} \\ \Delta\tilde{z}^{F} \end{bmatrix} + \begin{bmatrix} \Delta\tilde{y}^{F}\psi - \Delta\tilde{z}^{F}\theta \\ \Delta\tilde{z}^{F}\phi - \Delta\tilde{x}^{F}\psi \\ \Delta\tilde{x}^{F}\theta - \Delta\tilde{y}^{F}\phi \end{bmatrix} + \begin{bmatrix} 0 \\ \int_0^t (v_p^F + u_p^F\,\psi)dt \\ \int_0^t (w_p^F - u_p^F\,\theta)dt \end{bmatrix} \tag{7-20}$$

Equation (7-20) describes the relative position of the target and projectile resolved onto the fin-frame. It can be seen that this equation encapsulates a great deal of detail which would require a large amount of computation if the perturbation equations had not been developed.

It may be useful to identify the terms which appear on the right-hand side of equation (7-20), thus:

$\Delta\tilde{x}^F$, $\Delta\tilde{y}^F$, $\Delta\tilde{z}^F$	are relative position components determined during the reference trajectory.
u_p^F	projectiles component of velocity along the x-axis of the fin-frame.
v_p, w_p, θ, ψ	perturbations of the projectiles motion

Thus, $\Delta \tilde{x}^F$, $\Delta \tilde{y}^F$, $\Delta \tilde{z}^F$, u_p^F, are time-variant coefficients to be determined from the computation of a reference trajectory while v_1 w_p, θ, ψ are state-variable perturbations to be determined from a linear model of an interception.

The equations developed in this section are an alternative to the non-linear equations developed in sections (1) through (3) and in Section (6). All of these sections were developed with the description of motion referred to the aeroballistic coordinate frame. This convention has merit in facilitating understanding of complex motions. However, the aeroballistic frame convention does not permit a roll-perturbation, ϕ_F, to be derived as a small quantity of the first order. Formulating the description of the motion with respect to the fin frame permits retaining the roll perturbation as a small quantity of the first order in a linear model.

Equation (7-20) has been helpful in understanding the roll-control of the fin-frame. It will also be of considerable importance when optimal design formulations are undertaken.

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APPENDIX A
7-DOF PROJECTILE MODULE DESIGN DETAILS

A.1 INTRODUCTION

This appendix describes the details of a moculd which simulates seven degrees-of-freedom of a CHAMP projectile. The following modules are involved in this production:

8210	Acceleration due to gravity
57130	Non-dimensional terms with wind
9200	Atmosphere
57110	Aerodynamic derivatives
69050	Aerodynamic forces and moments
57030	Equations of motion
860	Direction cosines
870	Transformation and integration
69060	Controlled spinning projectile
69220	7-DOF projectile including seeker and canard deflection

Details of these modules are presented in the following sections of this appendix. Background material on the modular software system has been published in the CHAMP Phase I Final Report, Appendix B.

A.2 ACCELERATION DUE TO GRAVITY (Module #8210)

The force due to gravity can be expressed conveniently in the inertial coordinate frame. The components in this frame are:

$$\left.\begin{aligned} X_G^o &= 0 \\ Y_G^o &= 0 \\ Z_G^o &= mg \end{aligned}\right\} \quad \text{(where } g = \text{acceleration due to gravity)} \qquad \text{A-1}$$

These components can be transformed to the unprimed coordinate frame by the following transformation:

$$X_G = \mathbf{C}\, X_G^o \qquad \text{A-2}$$

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} mg \qquad \text{A-3}$$

$$\begin{bmatrix} X_G/m \\ Y_G/m \\ Z_G/m \end{bmatrix} = \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} g \qquad \text{A-4}$$

and equation (A-4) gives the terms which are required in the force equilibrium equations, (3-2).

This computation can be performed by Module No. 8130 which was coded directly in MARCO-10. The module is represented below and is summarized in Table A-1.

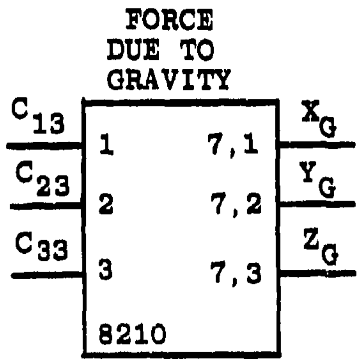


Figure A-1. Module No. 8210 Force Due to Gravity.

TABLE A-1

MODULE NO. 8210 - ACCELERATION DUE TO GRAVITY

L	M	Algebraic Symbol	Dimensions	Explanation
Connections				
3	1	c_{13}	dimensionless	direction cosine
	2	c_{23}	dimensionless	direction cosine
	3	c_{33}	dimensionless	direction cosine
Parameters				
4	1	mg	lbs	weight of the projectile
Intermediates				
7	1	X_G	lbs	force components due to gravity
	2	Y_G	lbs	
	3	Z_G	lbs	

A.3 NON-DIMENSIONAL TERMS (Modules #57000 & #57130)

The inputs to this module are:

z = the negative of the altitude, h (feet)
u,v,w = velocity components (feet/second)
p_F, p_B, q, r = angular velocity components (radians/second)

The functions of this module are:

(1) to transform z into h=-z, which is required as an input by the ATMOSPHERE module (#9200).

(2) to derive V, α, and β from u,v,w

(3) to derive the non-dimensional terms $p_F d/2v$, $p_B d/2v$, $qd/2v$, and $rd/2v$, where d is a reference length.

By definition:

$$V = \sqrt{u^2+v^2+w^2} \qquad \text{A-5}$$

$$\sin\alpha = \frac{w}{V} \; ; \; \tan\alpha = \frac{w}{\sqrt{u^2+v^2}} \; ; \; \alpha = \tan^{-1}\frac{w}{\sqrt{u^2+v^2}} \qquad \text{A-6}$$

$$\sin\beta = \frac{v}{\sqrt{u^2+v^2}} \; ; \; \tan\beta = \frac{v}{u} \; ; \; \beta = \tan^{-1}\frac{v}{u} \qquad \text{A-7}$$

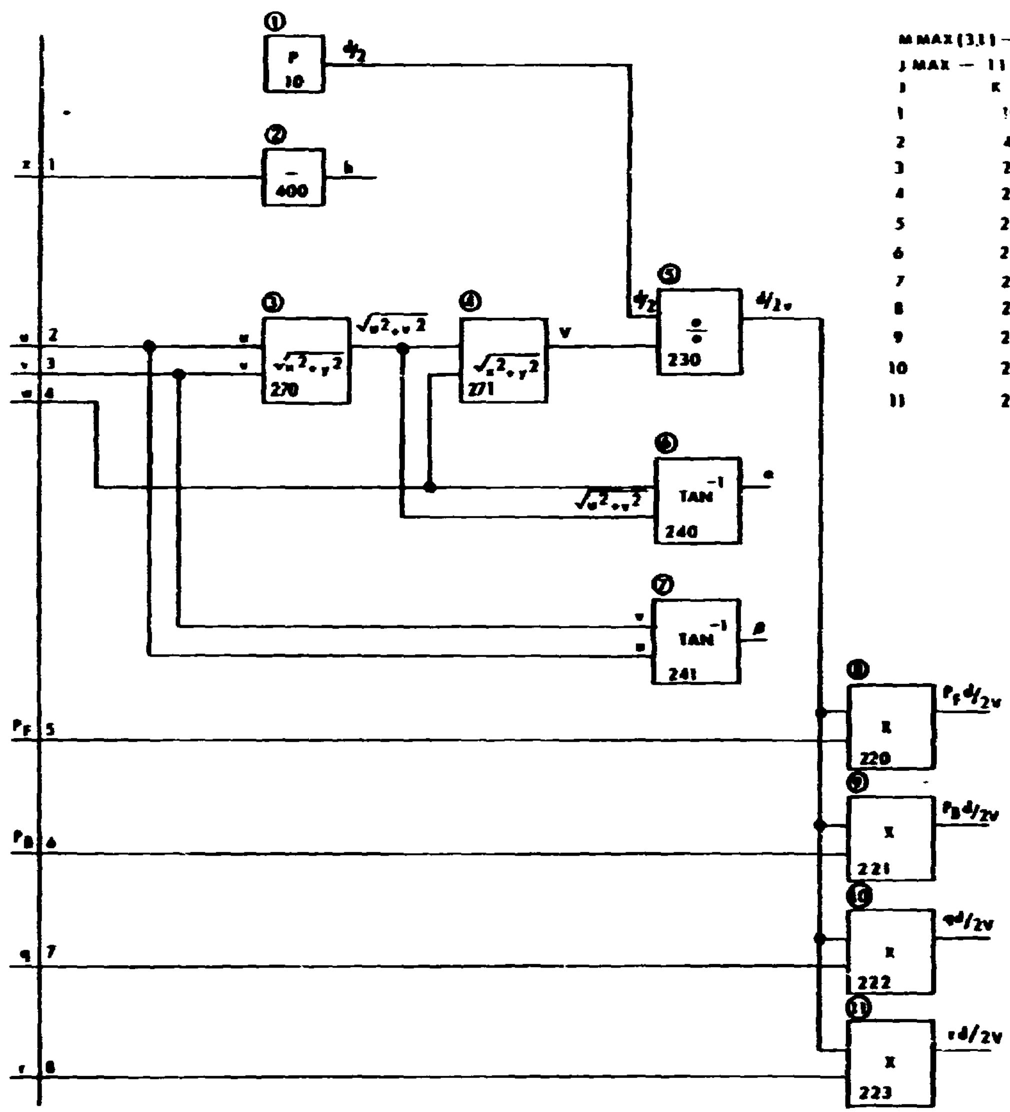
The composite module can be implemented with standard modules as shown in Figure A-2.

This same module can be employed to evaluate the non-dimensional terms required to describe the effect of winds, provided the following replacements are made:

$$\left.\begin{aligned} u &\leftarrow u+u_w \\ v &\leftarrow v+v_w \\ w &\leftarrow w+w_w \end{aligned}\right\} \qquad \text{A-8}$$

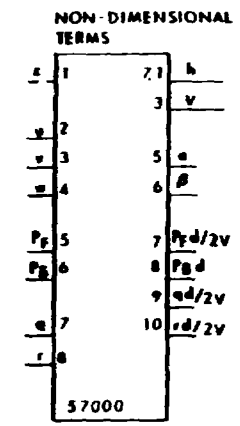
where u,v,w are state variables resolved on the unprimed coordinate frame. u_w, v_w, w_w are wind components resolved on the unprimed coordinate frame.

Now the wind components are best described in the inertial frame, so that:



M MAX (3,1) — 8
J MAX — 11

J	K
1	10
2	400
3	270
4	271
5	230
6	240
7	241
8	220
9	221
10	222
11	223



MODULE NUMBER 57000
NON-DIMENSIONAL TERMS

$$\begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} u_w^o \\ v_w^o \\ w_w^o \end{bmatrix} \qquad \text{A-9}$$

The transformation and additions have been combined in a single module as shown in Figure A-3 and Table A-2.

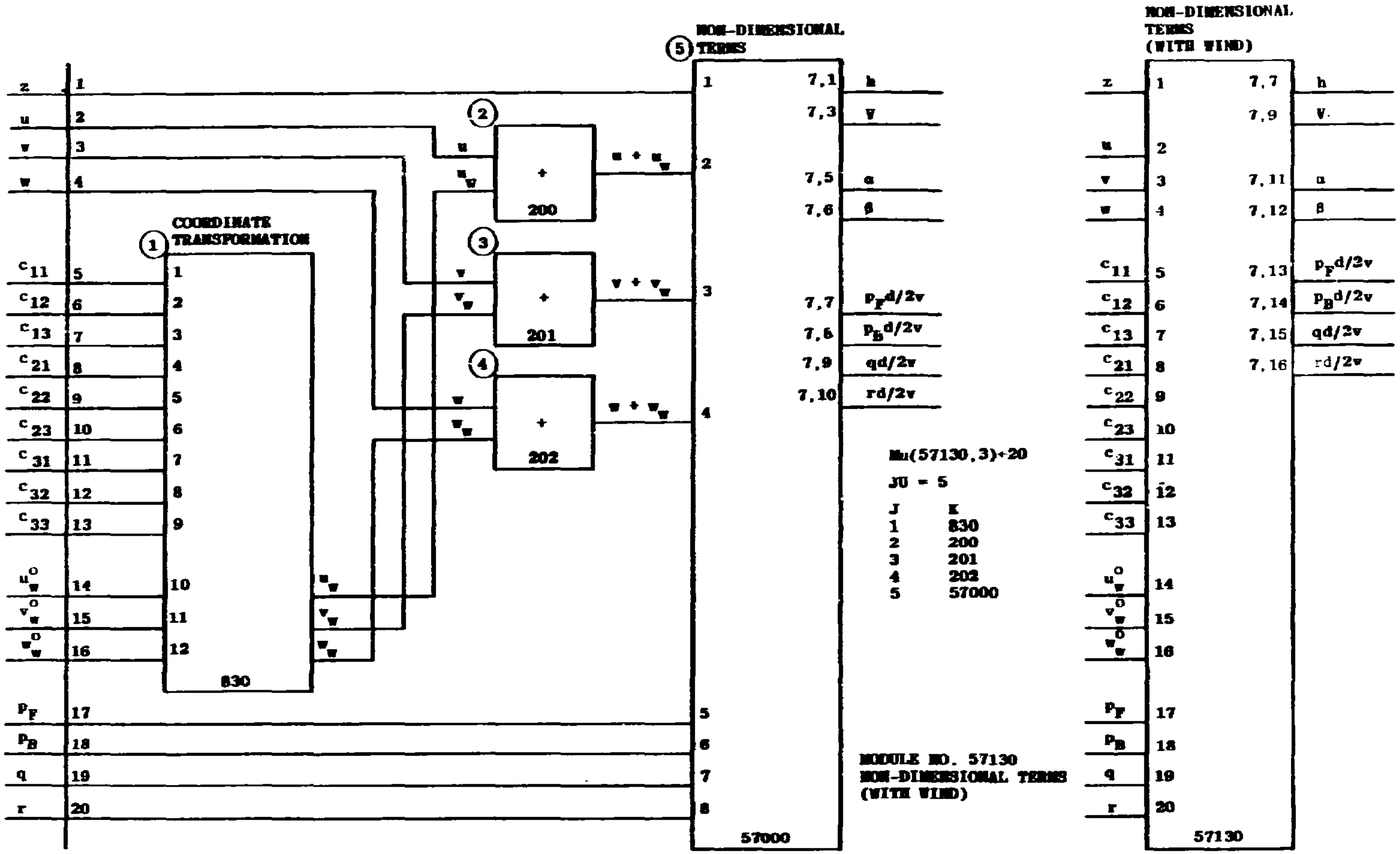


Figure A-3. Module No. 57130 Non-Dimensional Terms (with wind)

TABLE A-2

MODULE NO. 57130 NON-DIMENSIONAL TERMS (WITH WIND) DICTIONARY

L	M	Algebraic Symbol	Dimensions	Explanations
Connections				
3	1	z	ft	z position coordinate
	2	u	ft/sec	velocity components resolved onto the unprimed coordinate frame
	3	v		
	4	w		
	5-13	c_{ij}	dimensionless	direction cosines
	14	u_w^o	ft/sec	velocity components of the wind resolved onto the inertial coordinate frame
	15	v_w^o		
	16	w_w^o		
	17	p_F	rad/sec	x-component of angular velocity of the fin frame
	18	p_B	rad/sec	x-component of angular velocity of the body
	19	q	rad/sec	angular velocity components
	20	r		
Parameters				
4	1	$d/2$	ft	reference length
Intermediates				
7	1	u_w	ft/sec	velocity components of the wind resolved into the unprimed coordinate frame
	2	v_w		
	3	w_w		

TABLE A-2 (continued)

MODULE NO. 57130 NON-DIMENSIONAL TERMS (WITH WIND) DICTIONARY

L	M	Algebraic Symbol	Dimensions	Explanation
Intermediates				
	4	$u+u_w$	ft/sec	velocity components of the relative wind resolved onto the unprimed coordinate frame
	5	$v+v_w$	ft/sec	velocity components of the relative wind resolved onto the unprimed coordinate frame
	6	$w+w_w$	ft/sec	velocity components of the relative wind resolved onto the unprimed coordinate frame
	7	h	ft	altitude
	9	V	ft/sec	magnitude of the relative wind
	11	α	rad	angle-of-attack
	12	β	rad	side slip angle
	13	$p_F d/2v$	dimensionless	non-dimensional terms
	14	$p_B d/2v$	dimensionless	non-dimensional terms
	15	$qd/2v$	dimensionless	non-dimensional terms
	16	$rd/2v$	dimensionless	non-dimensional terms

A.4 ATMOSPHERE (MODULE NO. 9200)

The relationships necessary to define atmospheric properties as a function of altitude are based on standard equations of fluid statics. The numbers used in the equations are those of the 1959 ICAO Standard Atmosphere and its extension. The Model is piecewise continuous in three segments. It is based on a constant lapse rate from sea level to the tropopause (36,089 ft.). The atmosphere is assumed to be isothermal from 36,089 ft. to 82,021 ft. From 82,021 ft. to 154,199, the static temperature increases at a constant rate.

The outputs of the atmosphere model are uniquely determined given the altitude as an input. These outputs include static pressure ratio, and local speed of sound. In this particular formulation, one subsidiary calculations is also performed. The airspeed is used as an input to the model so that Mach No. can be calculated. Thus, having specified altitude and airspeed, the model determines static pressure ratio and Mach No. for use in calculating aerodynamic forces and torques. This module is coded in MARCO-10.

Definitions:

a	is local speed of sound in ft/sec
λ	is ratio of local static pressure to sea level static pressure
M	is Mach Number
tr	is the ratio of local static temperature to sea level static temperature
V	is true airspeed in ft/sec
h	is altitude above sea level in ft.

Connections:

V

h

Parameters: None.

Initial States: None.

Intermediates:

$$\left.\begin{aligned} tr &= 1-(6.8754 \times 10^{-6})h \\ \lambda &= tr^{5.2561} \end{aligned}\right\} \quad 0 \le h \le 36089$$

$$\left.\begin{aligned} tr &= 0.75187 \\ \lambda &= 1.2656e^{(-4.8063 \times 10^{-5})h} \end{aligned}\right\} \quad 36089 < h \le 82021$$

$$\left.\begin{aligned} tr &= 0.49160 + 3.1732 \times 10^{-6}h \\ \lambda &= 0.02456(0.65383 + (4.2204 \times 10^{-6})h)^{-11.388} \end{aligned}\right\} \quad 82021 < h \le 154199$$

$$\left.\begin{aligned} tr &= 1.0 \\ \lambda &= e^{(-.322004 + (-4.115414 \times 10^{-5})h)} \end{aligned}\right\} \quad h > 154199$$

$$a = 1116.9\sqrt{tr}$$

$$M = V/a$$

States: None.

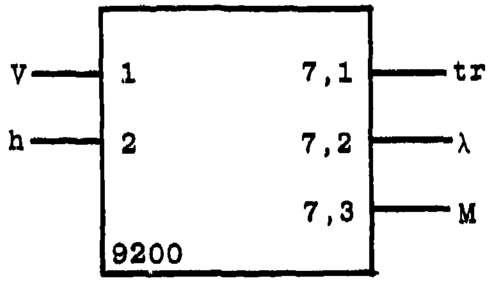


Figure A-4. Module No. 9200 - Atmosphere

A.5 AERODYNAMIC FORCES AND MOMENTS

The inputs to this module are:

a. The control deflections (δ_Y and δ_Z)
b. The cant angle (Δ_F)
c. The pressure ratio and Mach Number (λ and M)
d. The non-dimensional terms (α, β, $p_F d/2v$, $p_B d/2v$, $qd/2v$, $rd/2v$)

The most important outputs are:

a. The components of aerodynamic force (X, Y, Z)
b. The components of aerodynamic moment (L_F, L_B, M, N)

The simulation is synthesized in two steps. First, the aerodynamic derivatives are evaluated as functions of Mach Number. This is formulated in Modules Number 57110 and 8130. Then the derivatives are combined to evaluate the non-dimensional force and moment coefficients and are multiplied by the appropriate dimensional terms to yield force components in pounds and moment components in foot-pounds. This is accomplished in Module Number 69050.

A.5.1 AERODYNAMIC DERIVATIVES (Module No. 57110)

The several aerodynamic derivatives are expressed as quadratic functions of Mach Number; i.e.,

$$C = A_o + A_1 M + A_2 M^2$$

where the coefficients A_o, A_1, A_2 are fitted to the best available wind tunnel data or estimates in an independent procedure. The derivatives which are employed in the simulation are:

C_A axial force

$C_{Z\alpha}$ normal force due to angle of attack

$C_{Z\delta}$ normal force due to fin deflection

$C_{m\alpha}$ pitching moment due to angle of attack

C_{mq} damping in pitch

$C_{n\alpha p}$ Magnus moment

$C_{m\delta}$ pitching moment due to fin deflection

$C^F_{\ell p}$ roll damping of the fin

$C^F_{\ell\Delta}$ rolling moment due to fin cant

$C^B_{\ell p}$ roll damping of the body

$C^F_{\ell\beta\delta}$ dihedral torque on the canard frame due to fin deflection

$C^B_{\ell\Delta}\Delta_B$ body rolling moment due to cant angle

The computation also evaluates the dynamic pressure:

$$q = 1482.52M^2\lambda \ (\text{lbs/ft}^2)$$

These terms are evaluated in Module 57110 presented in Figure A-5.

A.5.2 AERODYNAMIC FORCES MOMENTS (Module No. 69050)

Module Number 69050 evaluates the dimensional forces and moments which act on the projectile. An additional aerodynamic term is required beyond those included in Module Number 57110 and this is added using Module Number 8130. Module 69050 is illustrated in Figure A-6.

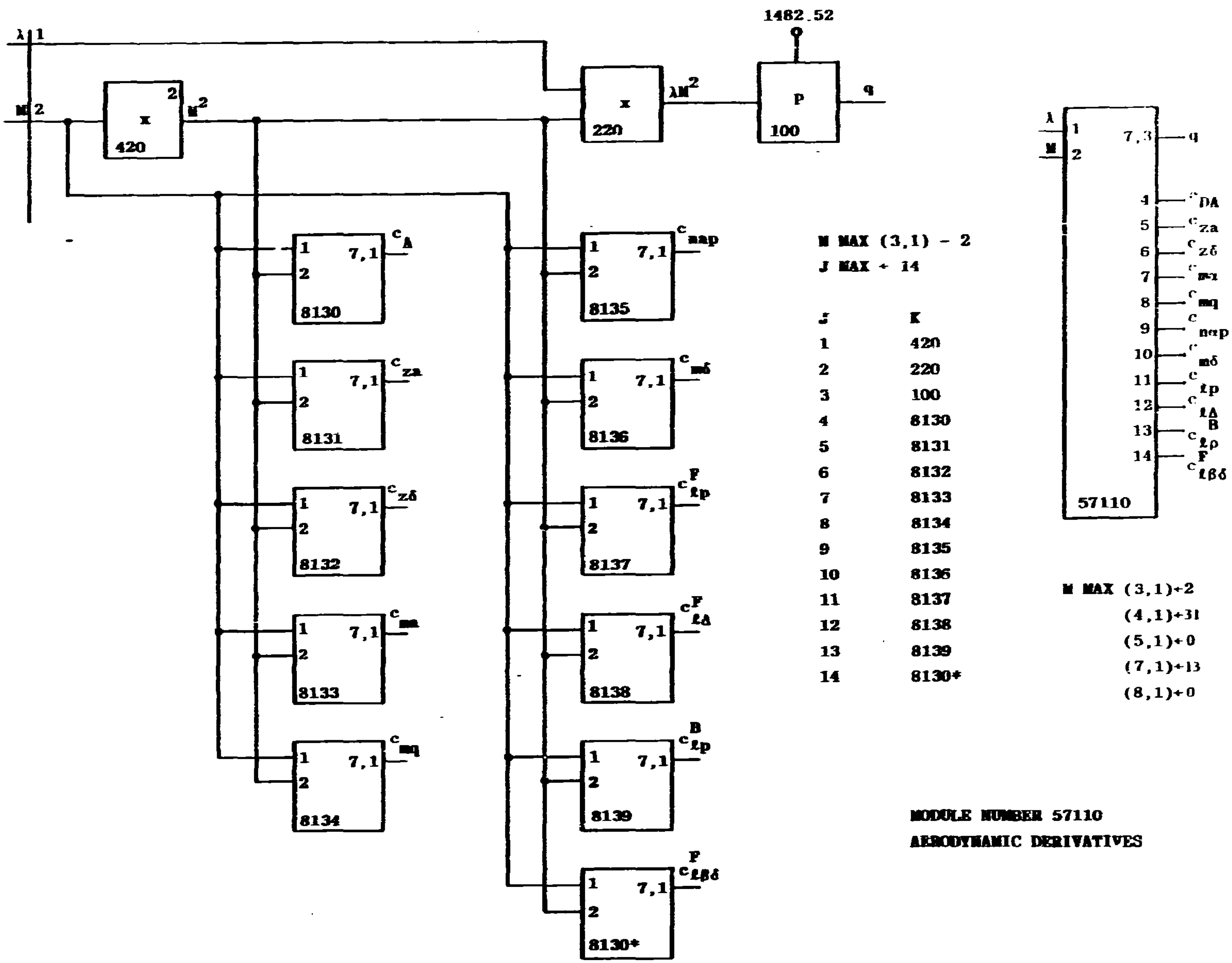
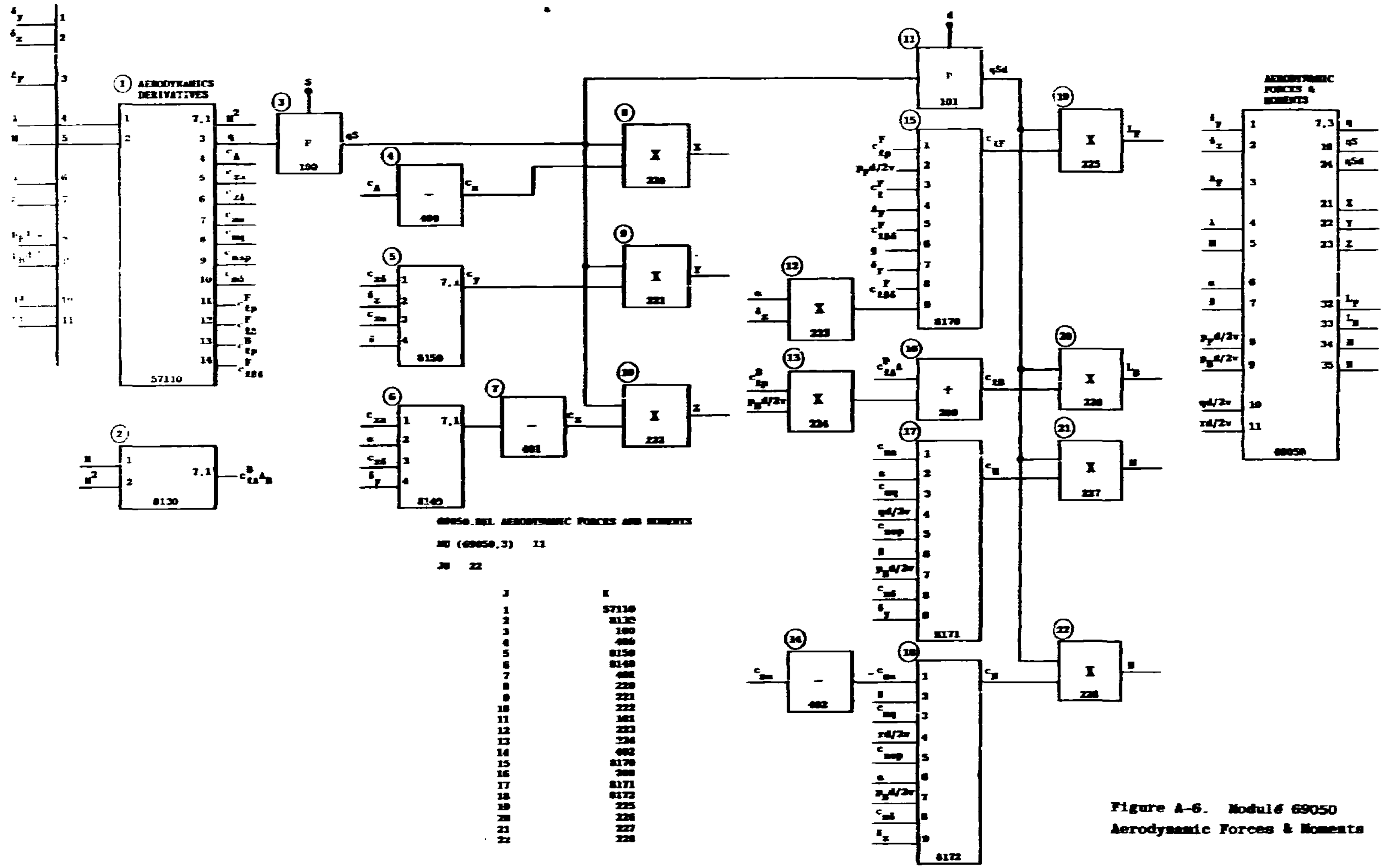


Figure A-5. Module Number 57110 Aerodynamic Derivatives



69050.MEL AERODYNAMIC FORCES AND MOMENTS

NO (69050,3) 11

JN 22

J	K
1	57110
2	8130
3	100
4	400
5	8150
6	8140
7	401
8	220
9	221
10	222
11	101
12	223
13	224
14	402
15	8170
16	300
17	8171
18	8172
19	225
20	226
21	227
22	228

Figure A-6. Module 69050 Aerodynamic Forces & Moments

A.6 CONTROLLED SPINNING PROJECTILE (Module 69060)

Module 69060 simulates the seven degrees-of-freedom of a controlled spinning projectile in a manner which is consistent with the theory derived in Section 3 of the main body of this volume.

The coordinate frame for the module is made to correspond to the pseudo-stability axes by setting p = [69060,3,5] equal to zero. The module accepts wind components relative to the inertial coordinate frame and it computes the projectiles motion relative to the inertial coordinate frame.

Attention should be called to the sign conventions for wind. The wind components have been taken as positive when they are in the direction of the negative coordinate axes. This permits the total relative wind components to be written as $u+u_w$, $v+v_w$, $w+w_w$, consistent with equations 3-21.

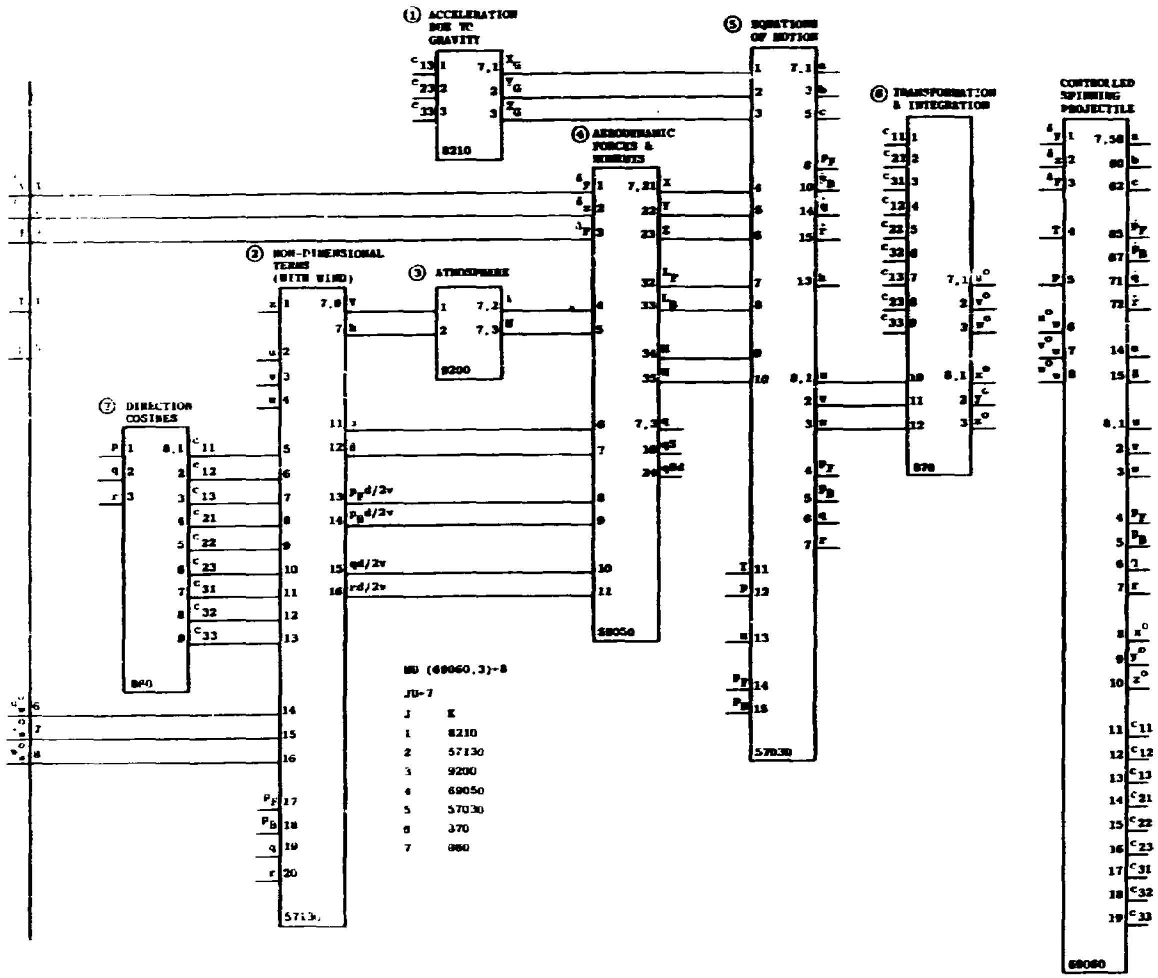


Figure A-7. Module 69060 Controlled Spinning Projectile

A.7 7-DOF PROJECTILE INCLUDING SEEKER AND CANARD FRAME (Module 69220)

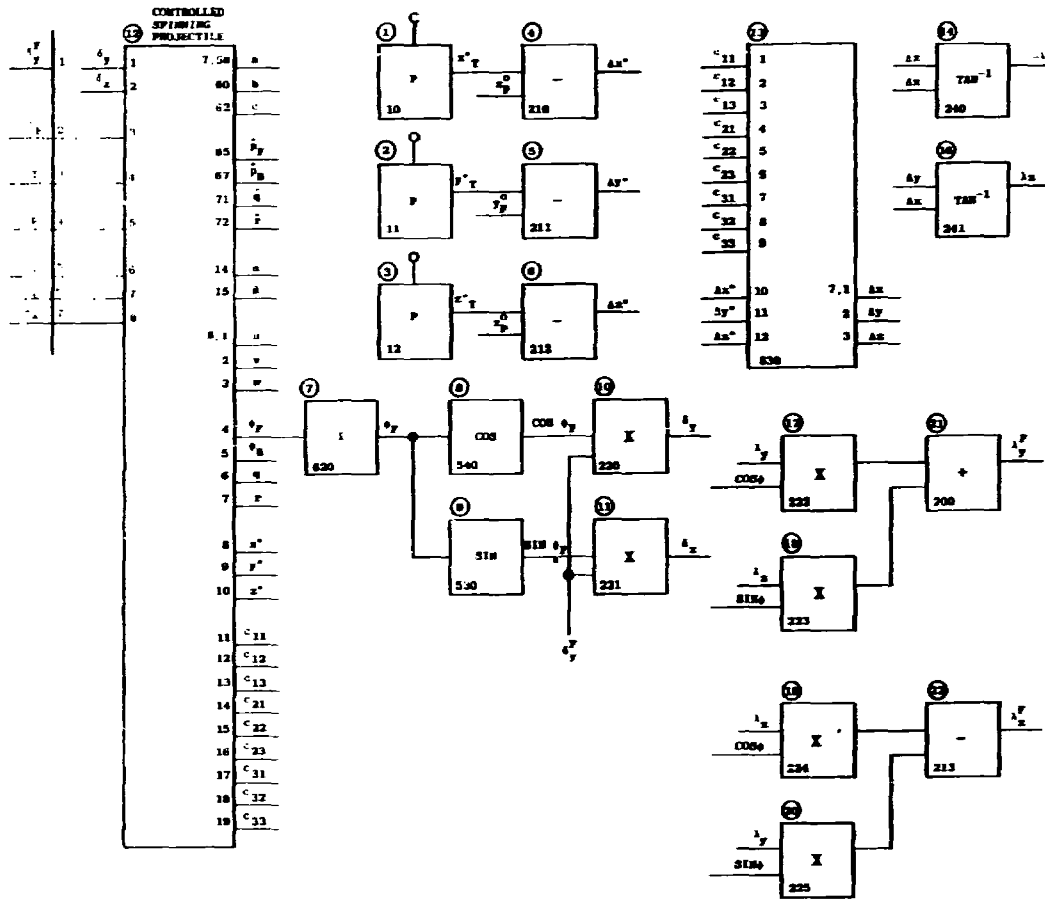
Module 69220 embodies the seven degree-of-freedom simulation of Module 69060. In addition, it includes the following features:

(1) It computes the relative position of the target and projectile in the inertial coordinate frame.

(2) The relative position is resolved onto the aeroballistic coordinate frame and look-angle components are evaluated.

(3) The angle, ϕ_F, between the fin-frame and the body is evaluated.

(4) The angle, ϕ_F, is used to resolve the canard deflection in the canard frame, δ_Y^F, onto the aeroballistic frame and the look-angle components onto the fin-frame.



MS(69220,3)-7

JU-22

J	K
1	10
2	11
3	12
4	210
5	211
6	212
7	620
8	540
9	530
10	220
11	221
12	65000
13	630
14	240
15	400
16	241
17	222
18	223
19	224
20	225
21	200
22	213

Figure A-8. Module 69220 7-DOF Projectile Including Seeker & Canard Deflection.

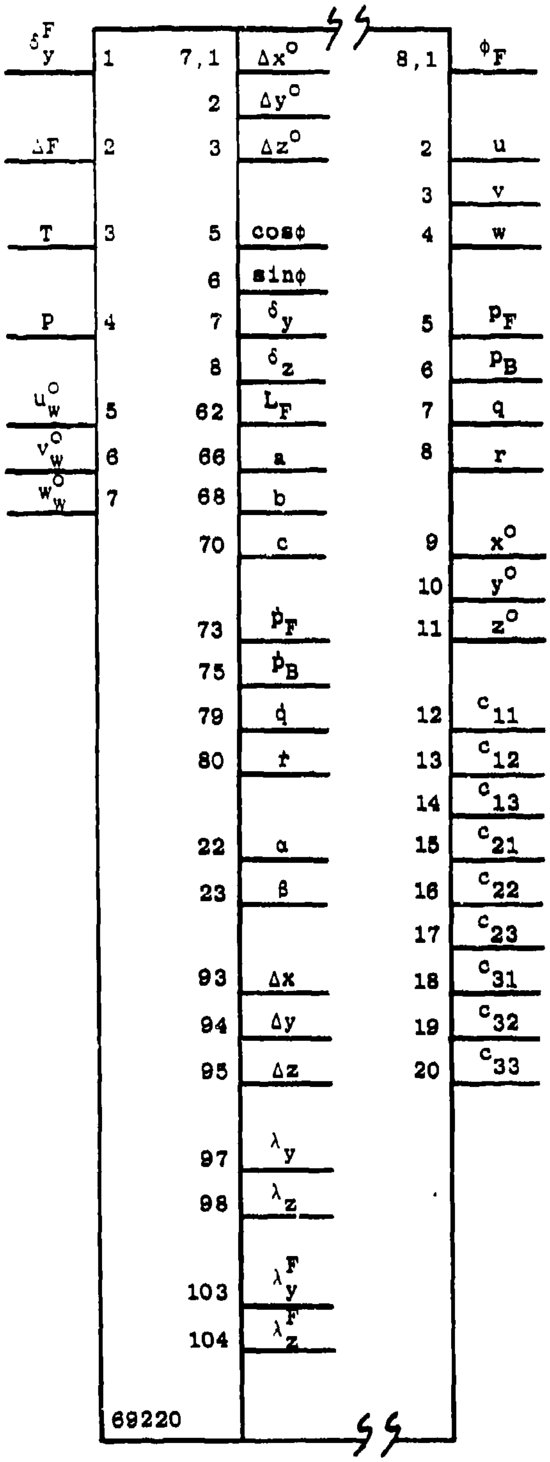


Figure A-9. Block Representation of Module 69220 7-DOF Projectile Including Seeker & Canard Deflection